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Theoretical High Energy Physics

Bachelor's Thesis

Deforming the $\mathfrak{su}(1,1)$ Phase Space Structure of Effective Loop Quantum Cosmology

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Abstract

Loop Quantum Cosmology (LQC) has been successful in quantizing various cosmological models by applying the ideas and techniques of Loop Quantum Gravity (LQG) to those models. The simplest one is the flat Friedmann-Robertson-Walker (FRW) model with a vanishing cosmological constant, coupled to a mass-less scalar field. When this model is quantized in LQC one obtains a consistent quantum model that replaces the Big Bang with a quantum bounce, resolving the singularity. From this quantum model, an effective Hamiltonian has been derived that describes the expectation values of the quantum operators classically. One can identify functions on the classical phase space that form an $\mathfrak{su}(1,1)$ Poisson algebra, and the effective (internal) Hamiltonian is precisely an element of this algebra. This provides us with the possibility to deform this algebra and thereby deform the Hamiltonian and the dynamics in the model. Guided by the fact that in three-dimensional LQG a deformation of the gauge group SU(2) can be used to introduce a cosmological constant in the theory, we investigate in this thesis the possibility that in the FRW-model under consideration a deformation of the $\mathfrak{su}(1,1)$ structure can be used to introduce the cosmological constant there.

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Contents

Conventions

- We use natural units: $c = \hbar = k = 1$, where c is the speed of light, $\hbar$ is the reduced Planck constant and k is Boltzmann's constant. Though in one case (section 3.1) we will write $\hbar$ explicitly to show its fundamental role.
- We adopt a $(-1, 1, 1, 1)$ metric signature.
- We adopt the summation convention: whenever an index appears both as subscript and as superscript in a single term in an equation, summation over the index is implied.

1 Introduction

Gravity is described excellently by Einstein's theory of General Relativity (GR). It seems, however, that reality ought to be described in a quantum mechanical way, as is the case in the standard model for the other forces of nature. The scale at which gravity is significant is, of course, much larger than the scale at which these other forces are at play; and we do not expect measurable quantum effects to be present at those large scales. However, when the theory is used to describe cosmological models, we encounter events in which quantum effects could become significant. The Big Bang, for instance, is such an event. Here the volume of the universe vanishes entirely and there really is no well-defined notion of space anymore. Moreover, quantities like the universe's energy density diverge. In these conditions we expect that quantum effects should become (hugely) significant, and therefore we should be careful trusting the predictions of GR around these events. To find out what really happens there we will need a quantum theory of gravity; a theory that describes gravity - or the geometry of spacetime - in a discrete way, using the framework of quantum mechanics. A promising candidate for such a theory is Loop Quantum Gravity (LQG) [1]. One of the important features of LQG is that it is background independent and non-perturbative, i.e. the metric is really treated as the field variable[1] and is quantized, instead of only its perturbations. Although LQG is not complete at present, it has succeeded in giving a consistent description of a kinematical Hilbert space corresponding to GR.

The ideas and techniques of LQG can also be applied to particular cosmological models, rather then to full GR, and this area of research is known as Loop Quantum Cosmology (LQC). In this way, several of these models have already been quantized successfully. A great achievement of LQC is that it is able to resolve the singularities, like the Big Bang, that are present in the classical models. The cosmological model that will be discussed in this thesis is the Friedmann-Robertson-Walker (FRW) model that is spatially flat ($k = 0$) and has a vanishing cosmological constant ($\Lambda = 0$), coupled to a mass-less scalar field. The classical solutions to this model show either a Big Bang singularity or a Big Crunch singularity, but when the model is quantized in LQC these are replaced by a *Quantum Bounce* as the universe reaches a non-vanishing minimum volume[2] (expectation value). This result has been obtained for generic semi-classical quantum states [2, 3, 4], i.e. states that are highly peaked on the classical trajectory far away from the bounce. And, of course, it is extremely unlikely that the quantum state of the universe would not be semi-classical, for then we would not experience the universe as we do. Therefore the quantum bounce is a profound and general result of LQC's description of this model. The model is sometimes called solvable Loop Quantum Cosmology (sLQC), for it is solvable analytically [5].

For semi-classical states, the dynamics of the expectation values of the quantum operators in the model can be described very accurately by making only a slight adjustment on the phase space of the classical model. This adjustment and its results are called the effective dynamics and it is valid for the FRW-model that is the subject of this thesis as well as for the analogous FRW-models in which *either* the spatial curvature is nonzero *or* the cosmological constant is nonzero. One can then identify functions on the classical phase space that form an $\mathfrak{su}(1,1)$ Poisson algebra, and for the ($k = 0,\ \Lambda = 0$) model, the (internal) Hamiltonian happens to be precisely an element of this

[1] Actually, the variables that are quantized are Holonomies and Fluxes, but these contain the same information as is contained in the metric.

[2] It would be more accurate to say that it is a representative cell in the universe that reaches this minimum volume, instead of the universe itself, for the volume of the whole universe can be infinite, depending on its topology. Yet, due to isotropy (which is the defining property of FRW-models), this representative cell is indeed representative of the whole universe.

Poisson algebra, as was already shown in [6]. In [7] this was used to carry out a group theoretical quantization of the model.
In this thesis we develop a general method to deform $\mathfrak{su}(1,1)$ Poisson algebras and we apply this method to the $\mathfrak{su}(1,1)$ algebra of phase space functions in effective sLQC. We then investigate if this deformation provides a way of introducing a nonzero cosmological constant in the model. This idea is motivated by results of the analysis of LQG in three dimensions, where it has been shown that a deformation of the gauge group SU(2) can be used to include a cosmological constant in the theory (see e.g. [8] for a summary). This has led to the claim that the same might be true in ordinary (four-dimensional) LQG and indeed there are some recent results that point in this direction. Recent papers on the subject are e.g. [9, 10]. Although the $\mathfrak{su}(1,1)$ Poisson algebra in our model is evidently not a gauge group as is SU(2) in LQG, it does define a structure on the model, and it is natural to wonder if a deformation of this structure could introduce interesting new dynamics, like the dynamics generated by a nonzero cosmological constant.

Structure of the Thesis

The structure of the thesis is as follows. In section 2 first of all the classical theory of gravity is summarized and this is applied to isotropic cosmological scenarios. In 2.3.2 the flat FRW-model with a vanishing cosmological constant, coupled to a scalar field - which is the model under consideration in this thesis - is discussed within this classical framework. In section 2.4 the Lagrangian and Hamiltonian formulation of GR are reviewed, the latter of which provides the basis for LQG and LQC. Section 3 then discusses briefly the matter of quantizing gravity.
In section 4 the phase space variables that are the starting point for LQC are introduced and in the model under consideration the Hamiltonian constraint (arising from the Hamiltonian formulation of GR) is reformulated in terms of these variables. Apart from some comments we do not review the quantization of this phase space. Instead, in section 4.2.1 we use the *effective dynamics* to describe the dynamics of the model with the leading quantum-modifications.
In section 5 we first review some group theoretical concepts needed to identify the structure on the phase space of the effective dynamics in LQC and then in 5.3 we propose a general procedure for deforming an $\mathfrak{su}(1,1)$ Poisson algebra. In section 6 we use the group theoretical framework to identify the $\mathfrak{su}(1,1)$ structure of the model, and we derive the evolution in the model, using this structure.
Finally, in section 7 we apply the developed deformation procedure to deform the $\mathfrak{su}(1,1)$ Poisson algebra on the phase space of effective sLQC. Doing this we obtain a deformed (internal) Hamiltonian leading to interesting new dynamics. This section concludes with a discussion of the results.

2 Classical Gravity and Cosmology

2.1 Newtonian Gravity

Newton was the one that (mathematically) unified the motion of the planets around the sun with the motion of all objects on earth. He showed that all bodies are subject to the same laws of motion, that are now known as Newton's laws of motion:

1. Every object in a state of uniform motion tends to remain in that state of motion unless an external force is applied to it.
2. The relationship between an object's mass m, its acceleration **a**, and the applied force **F** is
$$\mathbf{F} = m\mathbf{a} \ . \tag{2.1}$$
3. For every action there is an equal and opposite reaction.

And he stated that all bodies in the universe exert a gravitational attraction on each other. According to Newton, the gravitational force that a body of mass m_1 exerts on a body of mass m_2 is given by

$$\mathbf{F}_{12} = -\frac{Gm_1m_2}{r^2}\mathbf{e}_{12} \ , \tag{2.2}$$

where r is the distance between the two bodies, $\mathbf{e}_{12}$ is the unit vector pointing from body 1 to body 2 and G is the gravitational constant. In fact, today we know that (classically) this formula applies in the case where the two bodies are point masses. From there one can find the exact force between two bodies of any shape by integration. And in this way one can show that (2.2) is still exact for bodies that have a spherically symmetric mass distribution, like the sun and planets.

Later, Newtonian gravity has been reformulated as a field theory. The gravitational field is defined as the vector field $\mathbf{g}(\mathbf{r})$ such that a body of mass m at position $\mathbf{r}$ feels a force $m\,\mathbf{g}(\mathbf{r})$. This together with (2.2) implies that the gravitational field generated by a point mass m (in the origin) must be given by

$$\mathbf{g}(\mathbf{r}) = -\frac{Gm}{r^2}\mathbf{e}_{\mathbf{r}} \ , \tag{2.3}$$

where $r = |\mathbf{r}|$. Since gravity is a conservative force (or equivalently the curl of the gravitational field is zero) the gravitational field can be written as the gradient of a scalar potential Φ, that is called the gravitational potential,

$$\mathbf{g} = -\nabla\Phi \ . \tag{2.4}$$

Substituting this into Gauss's law,

$$\nabla \cdot \mathbf{g} = -4\pi G\rho \ , \tag{2.5}$$

where ρ is the matter density, one obtains Poisson's equation for gravity

$$\nabla^2\Phi = 4\pi G\rho \ . \tag{2.6}$$

The field theory formulation of Newtonian gravity resolved the problem of bodies separated in space interacting with each other: in the field theory formulation a body interacts solely with the gravitational field at its present location. However, we are still faced with the problem that when a field is generated, it travels instantaneously through all of space, which is impossible, according to Special Relativity. To resolve this, we need Einstein's theory of General Relativity.

2.2 General Relativity

General Relativity (GR) describes Gravity not as a force, but rather as a phenomenon (one could call it a fictitious force) arising due to the curvature of spacetime. This curvature is described by the Lorentz invariant infinitesimal spacetime interval, or line element, ds^2, defined as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \ . \tag{2.7}$$

Here we have used the summation convention, as we will continue to do from now on. $g_{\mu\nu}$ is called the metric tensor or just the metric. Actually $g_{\mu\nu}$ are just the components of the metric tensor as evaluated on a specific basis of one-forms[3], but most often $g_{\mu\nu}$ is just referred to as the metric. We also define the inverse metric $g^{\mu\nu}$ by $g^{\mu\nu} g_{\nu\sigma} = g_{\nu\sigma} g^{\mu\nu} = \delta^\mu{}_\sigma$. By parameterizing a path $x^\mu(\lambda)$ one can find the spacetime interval along the path by integration. According to this definition we distinguish three special types of paths in spacetime:

- Timelike paths: $ds^2 < 0$ everywhere along the path;
- Spacelike paths: $ds^2 > 0$ everywhere along the path;
- Lightlike paths: $ds^2 = 0$ everywhere along the path.

The proper time τ along a path in spacetime is the time that is measured by an observer that travels along the path. Infinitesimally we can write

$$d\tau^2 = -ds^2 = -g_{\mu\nu} dx^\mu dx^\nu \ . \tag{2.8}$$

Since observers always move along timelike paths, proper time is only defined for those paths.

The equivalence principle
A cornerstone in GR is the equivalence principle. The most basic (yet profound) version of it is known as the weak equivalence principle (WEP) and dates back to Galileo. He showed that objects falling freely[4] in the Earth's gravitational field all fall with the same acceleration, regardless of the mass of the objects. This turns out to be so because of the equivalence of gravitational mass and inertial mass. Classically, if m_i is the inertial mass of an object and m_g its gravitational mass, then the acceleration of the object is given by

$$\mathbf{a} = \frac{\mathbf{F}}{m_i} = \frac{m_g}{m_i} \mathbf{g} \ , \tag{2.9}$$

where $\mathbf{F}$ denotes the gravitational force and $\mathbf{g}$ the gravitational field. Galileo's observation was correct since

$$m_i = m_g \ , \tag{2.10}$$

which is a more modern formulation of the WEP. So the acceleration of a particle, regardless of its mass, is just the field value at the point. This is an essential feature of gravity and it has the

[3] A basis of one-forms is given precisely by the dx^μ as in (2.7), so it would also be accurate to say that the line element ds^2 *is* the metric.

[4] In GR the notion of an object falling freely means that no external force (other than gravity, which is really not a force) is acting on the object.

following consequence.
Consider a spaceship in outer space, where there is no gravitational field present, and consider the spaceship to be accelerating uniformly. All objects in the spaceship will then be pushed to the rear of the spaceship as if there were a force present. And when an object is thrown up[5], it will go up, and then 'fall' down to the back of the spaceship. And most importantly, any object will fall with the same acceleration, which is just the (opposite of the) acceleration of the spaceship. So the (fictitious) force has precisely the distinctive feature of gravity.
Although 'real' gravity grows stronger when you fall further down the gravitational field, this takes distance and time to notice. This led Einstein to postulate the Einstein equivalence principle (EEP). It can be stated as follows: *In small enough regions of spacetime, the laws of physics reduce to those of special relativity; it is impossible to detect a gravitational field by means of local experiments.* This provides the gateway from special relativity to general relativity.

The geodesic equation
GR generalizes Newton's first law to four-dimensional curved spacetime. Newtons first law states that 'every object in a state of uniform motion tends to remain in that state of motion unless an external force is applied to it'. In flat space 'to remain in a constant state of motion' means simply to move in a straight line. However, the generalization of a straight line in curved space is a geodesic: a curve that parallel transports its own tangent vector. The geodesic equation,

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0 \ , \tag{2.11}$$

describes these geodesic curves x^μ (parameterized by proper time τ). Here we have introduced the Christoffel connection, or Christoffel symbol,

$$\Gamma^\mu_{\rho\sigma} = \frac{1}{2}g^{\mu\nu}\left(\partial_\rho g_{\sigma\nu} + \partial_\sigma g_{\rho\nu} - \partial_\nu g_{\rho\sigma}\right) \ . \tag{2.12}$$

We should note that (2.11) is equivalent to

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0 \tag{2.13}$$

for any parameter λ that is related to τ by $\lambda = a\tau + b$. This is especially important for lightlike geodesics, for which the elapsed proper time is always zero and hence it can not be used to parameterize the curve.
In GR freely falling test particles follow geodesics. That is, particles with no forces acting on it follow trajectories described by the geodesic equation (2.13). In flat Minkowski space, where $g^{\mu\nu} = diag(-1, 1, 1, 1)$, the Christoffel symbol vanishes and we recover the equation of a straight line

$$\frac{d^2x^\mu}{d\lambda^2} = 0 \ . \tag{2.14}$$

Minimal-coupling principle
The minimal-coupling principle is a recipe for generalizing physics in flat spacetime to curved spacetime. It may be stated as follows:

[5]'up' meaning in the direction of acceleration.

1. Take a law of physics, valid in inertial coordinates in flat spacetime;
2. Write it in a coordinate invariant (tensorial) form;
3. Assert that the resulting law remains true in curved spacetime.

In practice this comes down to replacing the Minkowski metric $\eta_{\mu\nu}$ with a more general metric $g_{\mu\nu}$ and replacing the partial derivatives ∂_μ by covariant derivatives ∇_μ. (The covariant derivative is defined in appendix A.)
Using this principle we can actually derive that the equation of motion of a freely falling test particle is given by the geodesic equation. We start with the classical law

$$\frac{d^2x^\mu}{d\tau^2} = 0 \ . \tag{2.15}$$

This already looks like a coordinate invariant form, since x^μ is a well defined tensor. The second derivative of it with respect to τ, however, is not. Therefore we use the chain rule to write

$$\frac{d^2x^\mu}{d\tau^2} = \frac{d}{d\tau}\frac{dx^\mu}{d\tau} = \frac{dx^\nu}{d\tau}\frac{\partial}{\partial x^\nu}\frac{dx^\mu}{d\tau} = \frac{dx^\nu}{d\tau}\partial_\nu\frac{dx^\mu}{d\tau} = 0 \ . \tag{2.16}$$

This is a genuine tensorial expression and therefore we can now generalize the law to curved spacetime by replacing $\partial_\nu \to \nabla_\nu$. The resulting expression is

$$0 = \frac{dx^\nu}{d\tau}\nabla_\nu\frac{dx^\mu}{d\tau} \equiv \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} \ , \tag{2.17}$$

which is exactly the geodesic equation (2.11).

The Einstein field equations
Until now we have only seen the behavior of physics in a spacetime with a given metric. Now let's focus on what determines the metric. For this we will need the following definitions. The Riemann curvature tensor is defined as

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \ . \tag{2.18}$$

This tensor will vanish whenever there exists a frame in which the components of the metric are constant, i.e. the region of spacetime is flat. And the statement also works the other way round: when $R^\rho{}_{\sigma\mu\nu} = 0$, there exists a frame in which the metric components are constant[6]. From this we can define the Ricci tensor,

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} \ , \tag{2.19}$$

and the Ricci scalar, which is just the trace of the Ricci tensor:

$$R = R^\mu{}_\mu = g^{\mu\nu}R_{\mu\nu} \ . \tag{2.20}$$

The Einstein field equations (EFEs) are then given by

$$R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu} = 8\pi G T_{\mu\nu} \ , \tag{2.21}$$

[6] These two statements assume that the region of spacetime under consideration is simply-connected.

where $T_{\mu\nu}$ is the stress-energy tensor[7], which describes the matter in the universe. This tensor is conserved, i.e.

$$\nabla_\nu T^{\mu\nu} = 0 \ , \tag{2.22}$$

and this expresses the conservation of mass-energy in GR. There are several equivalent definitions for the energy-stress tensor, and for a scalar field one of these definitions is

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} \ , \tag{2.23}$$

where g is the determinant of the metric and S_M is the matter action (see section 2.4.1). The EFEs are General Relativity's analog to Poisson's equation for gravity (2.6) and they describe how the curvature of spacetime (gravity) is related to the distribution of matter and energy in the universe. Solving these equations means finding the components of the metric. The equations can be written equivalently as

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} T \, g_{\mu\nu} \right) \ . \tag{2.24}$$

From this expression it is easy to see that when the stress-energy tensor vanishes, the EFEs reduce to the vacuum EFEs,

$$R_{\mu\nu} = 0 \ . \tag{2.25}$$

2.3 FRW Cosmology

Cosmology is the study of the largest scale degrees of freedom in the universe, such as the universe's rate of expansion. To study those degrees of freedom one has to disregard most of the smaller degrees of freedom. This can be done by imposing symmetries on the universe.
Observations show that our universe is on large scales homogeneous and isotropic. Homogeneity is the property that there is no preferred point in the universe; all points are equivalent. And isotropy is the property that there is no preferred direction in the universe; all directions are equivalent. It can be shown that any metric satisfying these two properties can be written (in appropriate 'co-moving' coordinates (t, r, θ, ϕ)) as

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right) , \quad \text{where} \quad d\Omega^2 = d\theta^2 + sin^2(\theta) d\phi^2 \ . \tag{2.26}$$

In this expression $k = 0, \pm 1$ represents the curvature of 3-dimensional space. If $k = 1$, space has constant positive curvature, like the surface of a sphere. If $k = -1$, space has constant negative curvature. And if $k = 0$, space is flat. For a given k the metric is completely determined by the function $a(t)$, which is called the scale factor. The metric (2.26) is known as the Friedmann-Robertson-Walker (FRW) metric.
In cosmology matter is often modeled as a perfect fluid characterized by an energy density ρ, pressure P and 4-velocity u^μ. The stress-energy tensor for a perfect fluid is given in contravariant form by

$$T^{\mu\nu} = (\rho + P) \, u^\mu u^\nu + P g^{\mu\nu} \ , \tag{2.27}$$

[7]The stress-energy tensor is sometimes also called the energy-momentum tensor or even stress-energy-momentum tensor.

where we must have $u^\mu = (1, 0, 0, 0)$ due to homogeneity. By substituting (2.26) and (2.27) in the EFEs (2.21) one obtains the Friedmann equations,

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho \ , \tag{2.28}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) \ . \tag{2.29}$$

The former is known as the first Friedmann equation and the latter as the second Friedmann equation. Alternatively (2.28) is often called *the* Friedmann equation and (2.29) is sometimes called the acceleration equation. Combining the two gives us a third (not independent, but nevertheless important) equation

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + P) \ , \tag{2.30}$$

that expresses the conservation of energy in a FRW universe. In fact, this equation is the ($\mu = 0$) component of (2.22) in a FRW-universe. Any two of the equations (2.28), (2.29), (2.30) are equivalent to any other two, so to solve them one may consider any two.
One needs, however, one more ingredient to solve the Friedmann equations: an equation of state, i.e. an equation that relates P and ρ. In cosmology often the equation of state is considered to be that of a barotropic perfect fluid with $P = w\rho$ for a constant w. In that case (2.30) can be rewritten as

$$\dot{\rho} = -3\frac{\dot{a}}{a}(1 + w)\rho \ , \tag{2.31}$$

and this can be directly integrated to

$$\rho = \rho_0 \left(\frac{a}{a_0}\right)^{-3(1+w)} \ . \tag{2.32}$$

Then one only needs to substitute this in the Friendmann equation (2.28) and solve it for $a(t)$.

2.3.1 The Cosmological Constant

Up to this point we have not discussed the cosmological constant. Yet it is an important ingredient in GR. The cosmological constant was originally introduced by Einstein when he discovered that he could modify his field equations (2.21) to contain an extra term, while maintaining a consistent theory, but making it possible to construct cosmological solutions of static universes. He did not like the idea of an expanding or contracting universe. The modified field equations read

$$R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \ , \tag{2.33}$$

where Λ is the cosmological constant, which can be any real number, in principle. Current observations suggest that our universe contains a positive cosmological constant which is very small but nonzero. Often Λ is absorbed in the definition of the stress-energy tensor, and in that case the modified EFEs are simply the original ones (2.21). More precisely, if we consider a FRW-model, then the cosmological constant can be absorbed in the definition of the energy density and pressure of the matter in the universe by making the replacements

$$\rho \to \rho + \frac{\Lambda}{8\pi G} \ , \tag{2.34}$$

$$P \to P - \frac{\Lambda}{8\pi G} \ . \tag{2.35}$$

In this way we see that the cosmological constant attributes an energy density and a pressure to the vacuum.

2.3.2 The (k=0, Λ=0) Mass-less Scalar Field Solution

We will now use the laws of General Relativity to analyze classically the model that is the main subject of this thesis: the FRW-model with $k = 0$ (the universe is spatially flat) and $\Lambda = 0$ (we do not have to redefine the energy density and pressure of the matter), coupled to a scalar field ϕ.

The scalar field

The stress-energy tensor of a scalar field is given in covariant form by

$$T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - \tfrac{1}{2}g_{\mu\nu}g^{\rho\sigma}\partial_\rho\phi\,\partial_\sigma\phi - g_{\mu\nu}V(\phi) \ . \tag{2.36}$$

Imposing homogeneity on the scalar field, due to which all spatial derivatives vanish, and using that $g^{00} = -1$, we have

$$T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi + \tfrac{1}{2}g_{\mu\nu}\dot{\phi}^2 - g_{\mu\nu}V(\phi) \ , \tag{2.37}$$

which can be written in mixed form as

$$T^\mu{}_\nu = g^{\mu\alpha}\partial_\alpha\phi\,\partial_\nu\phi + \tfrac{1}{2}\delta^\mu{}_\nu\dot{\phi}^2 - \delta^\mu{}_\nu V(\phi) \ . \tag{2.38}$$

Then the nonzero entries are

$$T^0{}_0 = -\tfrac{1}{2}\dot{\phi}^2 - V(\phi) \ , \quad T^1{}_1 = T^2{}_2 = T^3{}_3 = \tfrac{1}{2}\dot{\phi}^2 - V(\phi) \ . \tag{2.39}$$

Now we write the stress-energy tensor of a perfect fluid (2.27) in mixed form as well,

$$T^\mu{}_\nu = (\rho + P)\,u^\mu u_\nu + P\,\delta^\mu{}_\nu \ , \tag{2.40}$$

and using $u^\mu = (1, 0, 0, 0)$ we calculate the nonzero entries

$$T^0{}_0 = -\rho \ , \quad T^1{}_1 = T^2{}_2 = T^3{}_3 = p \ . \tag{2.41}$$

Equating the corresponding components in (2.39) and (2.41) yields

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \ , \qquad P = \frac{1}{2}\dot{\phi}^2 - V(\phi) \ . \tag{2.42}$$

For a mass-less scalar field, we have $V(\phi) = 0$, and hence we have simply

$$\rho = P = \frac{1}{2}\dot{\phi}^2 \ . \tag{2.43}$$

Solution to the Friedmann equations

Substituting ρ and P in the Friedmann equations (2.28), (2.29) and setting $k = 0$ we get two differential equations for two unknown functions,

$$\ddot{a} = -\frac{8\pi G}{3} a\dot{\phi}^2 \ , \tag{2.44}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{4\pi G}{3}\dot{\phi}^2 \ ,. \tag{2.45}$$

and we can solve these analytically. We begin by substituting former in the latter and eliminating $\dot{\phi}^2$. This gives us

$$\left(\frac{\dot{a}}{a}\right)^2 = -\frac{1}{2}\frac{\ddot{a}}{a} \quad \Rightarrow \quad 2\dot{a}^2 = -\ddot{a}a \ . \tag{2.46}$$

Now we define $v = \frac{da}{dt} \equiv \dot{a}$ so that we can write

$$\ddot{a} = \frac{dv}{dt} = \frac{dv}{da}\frac{da}{dt} = v\frac{dv}{da} \ , \tag{2.47}$$

and we rewrite (2.46) in terms of v and solve for v:

$$2v^2 = -\frac{dv}{da}va \tag{2.48}$$

$$\frac{dv}{da} = -2\frac{v}{a} \tag{2.49}$$

$$\frac{dv}{v} = -2\frac{da}{a} \tag{2.50}$$

$$ln(v) = -2ln(a) + C_1 = ln(C_1 a^{-2}) \tag{2.51}$$

$$v = \frac{C_1}{a^2} \ , \tag{2.52}$$

where C_1 is an integration constant. Now we can substitute back $v = \frac{da}{dt}$ and solve for $a(t)$:

$$v = \frac{da}{dt} = \frac{C_1}{a^2} \tag{2.53}$$

$$\frac{a^2}{C_1}da = dt \tag{2.54}$$

$$t = \frac{a^3}{3C_1} + C_2 \tag{2.55}$$

$$a^3 = 3C_1(t - C_2) \tag{2.56}$$

$$a(t) = \mathrm{sgn}\left(3C_1(t - C_2)\right)|3C_1(t - C_2)|^{\frac{1}{3}} \tag{2.57}$$

$$a(t) = C_1 \mathrm{sign}\left(t - C_2\right)|t - C_2|^{\frac{1}{3}} \ , \tag{2.58}$$

where C_1 and C_2 are integration constants and in the last step C_1 has been rescaled. Because the scale factor should always be positive, we conclude that there are two non-trivial solutions:

$$C_1 > 0: \quad a(t) = a_m(t - t_0)^{1/3} \quad \text{on the domain } (t > t_0) \ , \tag{2.59}$$

$$C_1 < 0: \quad a(t) = a_m(t_0 - t)^{1/3} \quad \text{on the domain } (t < t_0) \ , \tag{2.60}$$

for a reference time t_0 and positive constant a_m. The first solution describes a universe that starts with a Big Bang and keeps expanding forever. The latter solution describes the exact opposite; it describes a universe that has always been contracting and eventually reaches a Big Crunch. We can now substitute our solutions for $a(t)$ in (2.45) to obtain the evolution of the scalar field ϕ:

$$\dot{\phi}^2 = \frac{1}{12\pi G}\frac{1}{(t-t_0)^2} \tag{2.61}$$

$$\dot{\phi} = \pm\frac{1}{\sqrt{12\pi G}}\frac{1}{(t-t_0)} \tag{2.62}$$

$$\phi(t) - \phi(t_1) = \pm\frac{1}{\sqrt{12\pi G}} ln\left(\frac{t-t_0}{t_1-t_0}\right) \ . \tag{2.63}$$

Furthermore we compute the following quantities,

$$\rho(t) = P(t) = \tfrac{1}{2}\dot{\phi}^2 = \frac{1}{24\pi G}\frac{1}{(t-t_0)^2} \ , \tag{2.64}$$

that tend to infinity when t tends to t_0. Thus we encounter a singularity at $t = t_0$, which is either a Big Bang singularity or a Big Crunch singularity, depending on the chosen solution of $a(t)$. In this event the classical theory is not valid anymore and quantum effects should be included in the theory to find out what really happens here. In section 4 we will see how the dynamics is modified, when the model is described from the perspective of LQC. To be able to compare the classical dynamics with the quantum dynamics later, we now write the volume V of a certain cell $\mathcal{V}$ as a function of the scalar field ϕ, using that $V \propto a(t)^3$. Moreover, we set $t_0 = 0$. We then obtain the relation

$$V(\phi) = V_0 e^{\pm\sqrt{12\pi G}(\phi-\phi_0)} \tag{2.65}$$

for some real constant ϕ_0 and some positive constant V_0.

2.4 Lagrangian and Hamiltonian Formulation of GR

2.4.1 Lagrangian Formulation

Most prescriptions for quantizing a classical theory require that the theory is formulated as a Lagrangian or Hamiltonian system. Therefore we would like to have GR formulated as such a system. In this section I will summarize the Lagrangian and Hamiltonian formulation of GR. For more elaboration on the Langrangian formulation I refer the reader to [11] and for more elaboration on the Hamiltonian formulation I refer the reader to [12].

First we consider geometry (gravity) alone; the case in which there is no matter involved. Then the dynamics of GR is provided by the Hilbert action,

$$S_H = \int \mathcal{L}_g d^4x = \int \sqrt{-g} R d^4x, \quad \text{where} \quad \mathcal{L}_g = \sqrt{-g}R \tag{2.66}$$

is the corresponding Lagrangian[8], that I will refer to as the Hilbert Lagrangian. The inverse metric $g^{\mu\nu}$ is used as field variable and the action is varied with respect to it. Then one finds that for

[8] Actually, its the Langrangian *density*, but more often than not it is just called the Lagrangian.

stationary points of the action we have[9]

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 0 \ , \tag{2.67}$$

which are the EFEs in vacuum. When we include matter, in the form of a scalar field, the action is modified to

$$S = \frac{1}{16\pi G} S_H + S_M \ , \tag{2.68}$$

where S_m is the action of the scalar field. Again varying this action with respect to the inverse metric one obtains that the stationary points satisfy[10]

$$\frac{1}{16\pi G}\left(R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu}\right) + \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} = 0 \ . \tag{2.69}$$

Recalling the definition of the stress-energy tensor of a scalar field (2.23) we recover the EFEs,

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 8\pi G\, T_{\mu\nu} \ . \tag{2.70}$$

Therefore in the case of a scalar field this Lagrangian formulation is equivalent to the EFEs of general relativity.

2.4.2 Hamiltonian Formulation

A Hamiltonian formulation of GR requires the breakup of space and time. Since this breakup can be different for different observers (coordinate systems) in GR, we need to specify what we mean by space and what we mean by time. We choose a time function t and a vector field t^μ on spacetime such that the surfaces Σ_t of constant t are spacelike Cauchy surfaces[11] and such that $t^\mu \nabla_\mu t = 1$. We then define the lapse function N and the shift vector N^a as

$$N = -t^\mu n_\mu = -g_{\mu\nu} t^\mu n^\nu, \qquad N^a = h^a_b t^b \ , \tag{2.71}$$

where n_μ is the unit vector perpendicular to Σ_t and $h_{ab} = g_{ab} + n_a n_b$ is the induced spatial metric on Σ_t. Here greek indices (μ, ν, …) are used for four-vectors on spacetime, whereas latin indices (a, b, …) are used for three-dimensional vectors on Σ_t. We use as field variables h_{ab}, N and the covariant form of the shift vector $N_a = h_{ab} N^b$. In these three values the same information is contained as in g^{ab} that we used as field variable in the Lagrangian approach. Our first job is to express the Hilbert Lagrangian (2.66) in terms of these variables. We have

$$\sqrt{-g} = N\sqrt{h} \quad \text{and} \quad R = {}^{(3)}R + K_{ab}K^{ab} - K^2 \ , \tag{2.72}$$

where K_{ab} is the extrinsic curvature of Σ_t, $K = K^a{}_a$ is the trace of K_{ab}, ${}^{(3)}R$ is the scalar curvature on Σ_t and h is the determinant of h_{ab}. Furthermore, discarding all terms that will give rise to boundary terms, we can write

$$K_{ab} = \frac{1}{2N}\left(\dot{h}_{ab} - D_a N_b - D_b N_a\right) \ , \tag{2.73}$$

[9] Assuming that the variation of the metric and its first derivative vanish at infinity.

[10] See previous footnote.

[11] A Cauchy surface is a subset of space-time which is intersected by every inextensible, non-spacelike (i.e. causal) curve exactly once.

where D_a is the covariant derivative operator on Σ_t associated with h_{ab} (defined in appendix A). Putting it all together, we can write the Hilbert Lagrangian as

$$\mathcal{L}_g = \sqrt{-g}R = N\sqrt{h}\left({}^{(3)}R + K_{ab}K^{ab} - K^2\right). \tag{2.74}$$

The momentum conjugate to h_{ab} then reads

$$\pi^{ab} = \frac{\delta \mathcal{L}}{\delta \dot{h}_{ab}} = \sqrt{h}\left(K^{ab} - Kh^{ab}\right). \tag{2.75}$$

The conjugate momenta to N and N_a are identical to zero, since no time derivatives of these variables appear in the Lagrangian. This tells us that N and N_a should not be viewed as dynamical variables. In fact, they play the role of Lagrange multipliers. So our configuration space is now just the space of Riemannian metrics h_{ab} on Σ_t. Again discarding boundary terms, the Hamiltonian density is then given by

$$\begin{aligned}\mathcal{H} &= \pi^{ab}\dot{h}_{ab} - \mathcal{L}_g \\ &= h^{1/2}\left\{N\left[-{}^{(3)}R + h^{-1}\pi^{ab}\pi_{ab} - \frac{1}{2}h^{-1}\pi^2\right] - 2N_b\left[D_a\left(h^{-1/2}\pi^{ab}\right)\right]\right\},\end{aligned} \tag{2.76}$$

where $\pi = \pi^a{}_a$. The variation of H with respect to N and N_a yields the constraints

$$\mathcal{C}^{grav} := -\sqrt{h}\,{}^{(3)}R + \frac{\pi^{ab}\pi_{ab}}{\sqrt{h}} - \frac{\pi^2}{2\sqrt{h}} = 0\ , \tag{2.77}$$

$$\mathcal{C}_b^{grav} := -2h^{1/2}D_a\left(h^{-1/2}\pi^{ab}\right) = 0\ , \tag{2.78}$$

that are known respectively as the Hamiltonian constraint and the diffeomorphism constraint. We see that we can write the Hamiltonian density as

$$\mathcal{H} = N\mathcal{C}^{grav} + N^a\mathcal{C}_a^{grav}\ , \tag{2.79}$$

and this implies that the Hamiltonian density is identical to 0. The variation of H with respect to h_{ab} yields

$$\dot{h}_{ab} = \frac{\delta H}{\delta \pi^{ab}} = 2h^{-1/2}N\left(\pi_{ab} - \frac{1}{2}h_{ab}\,\pi\right) + D_aN_b + D_bN_a, \tag{2.80}$$

$$\begin{aligned}\dot{\pi}^{ab} = \frac{\delta H}{\delta h_{ab}} = &-Nh^{1/2}\left({}^{(3)}R^{ab} - \frac{1}{2}{}^{(3)}Rh^{ab}\right) \\ &+ \frac{1}{2}Nh^{-1/2}h^{ab}\left(\pi_{cd}\pi^{cd} - \frac{1}{2}\pi^2\right) \\ &- 2Nh^{-1/2}\left(\pi^{ac}\pi_c{}^b - \frac{1}{2}\pi\pi^{ab}\right) \\ &+ h^{1/2}\left(D^aD^bN - h^{ab}D^cD_cN\right) \\ &+ h^{1/2}D_c\left(h^{-1/2}N^c\pi^{ab}\right).\end{aligned} \tag{2.81}$$

Those are known as the dynamical equations. The set of equations (2.77 - 2.81) is equivalent to the vacuum EFEs.

If there is matter present, a matter term is added to each constraint:

$$\mathcal{C} = \mathcal{C}^{grav} + \mathcal{C}^{matt} = 0 \ , \tag{2.82}$$

$$\mathcal{C}_a = \mathcal{C}_a^{grav} + \mathcal{C}_a^{matt} = 0 \ , \tag{2.83}$$

and we can write the resulting Hamiltonian density as

$$\mathcal{H} = N\left(\mathcal{C}^{grav} + \mathcal{C}^{matt}\right) + N^a\left(\mathcal{C}_a^{grav} + \mathcal{C}_a^{matt}\right) \ . \tag{2.84}$$

In the case of the FRW-model that we have already discussed classically and that we will discuss in the context of LQC, the diffeomorphism constraint is automatically satisfied. Therefore we can write the Hamiltonian as[12]

$$C = \int_\Sigma d^3x N\left(\mathcal{C}^{grav} + \mathcal{C}^{matt}\right) = N\left(C^{grav} + C^{matt}\right) \ . \tag{2.85}$$

Here $C^{grav} = \int_\Sigma d^3x\, \mathcal{C}^{grav}$ and $C^{matt} = \int_\Sigma d^3x\, \mathcal{C}^{matt}$. (The lapse function can be taken out of the integral due to homogeneity.) Moreover, in our case, where matter is described by a scalar field, C^{matt} is precisely the Hamiltonian of the scalar field, and we will use this explicitly later.

[12]Provided the integrals do not diverge; otherwise this may get tricky.

3 Quantizing Gravity and Cosmology

In the past century quantum mechanics has proven to be incredibly successful, and it seems that any fundamental theory of physics should be formulated within the framework of quantum mechanics. This has already been accomplished for the theories of all fundamental forces, except for the theory of gravity. E.g. the electromagnetic interaction and the strong interaction are described quantum mechanically in the standard model by quantum electrodynamics (QED) and quantum chromodynamics (QCD), respectively. Gravity, however, has not yet been formulated successfully as a quantum theory. And there are various reasons for this. The most important of these might be that quantum theories normally assume a fixed metric (most often the Minkowski metric), that serves as a background on which the dynamical quantum fields of the theory are defined. In General relativity, however, there is no fixed metric, since the metric is itself the dynamical field variable. We also saw in section 2.4.2 that the Hamiltonian formulation of GR leads to several constraints due to which the Hamiltonian (that formally can not be called a Hamiltonian anymore) vanishes entirely, and this leads to complications for the canonical quantization approach (see below). Because of these and other difficulties it is quite challenging to construct a quantum theory of gravity. Yet there are several promising approaches, one of which is Loop Quantum Gravity (LQG).

3.1 Canonical Quantization

One of the most frequently used quantization procedures is the canonical quantization procedure. The word *canonical* refers to the fact that the procedure is used to quantize theories that are formulated as Hamiltonian systems. From that starting point the procedure prescribes that one ought to replace any phase space variables $A, B, \ldots$ by operators $\hat{A}, \hat{B}, \ldots$ such that the commutators of the operators correspond to the Poisson brackets of the classical variables in the following way,

$$[\hat{A}, \hat{B}] = i\hbar \widehat{\{A, B\}} \; , \tag{3.1}$$

which is sometimes called the Dirac rule, for he was the one that proposed it.
The most basic example to illustrate this procedure is the case of a single non-relativistic free particle. In this case the phase space is described by the variables x and p that denote position and momentum of the particle, respectively. They satisfy $\{x, p\} = 1$ and therefore the canonical quantization says that in the quantum theory of this system one should have the commutation relation $[\hat{x}, \hat{p}] = i\hbar$, which is indeed the case in ordinary quantum mechanics. And the operators can thus be represented by the usual operators $\hat{x} = x, \; \hat{p} = -i\hbar\frac{\partial}{\partial x}$.

Remark. Altough we have adopted the convention $\hbar = 1$, in the expressions of this section we have written $\hbar$ explicitly, to show it's fundamental role in the canonical quantization procedure. In the following sections we will set it to one again and not write it explicitly anymore.

4 Loop Quantum Cosmology

The ultimate way to obtain a quantum theory of cosmology would be first to quantize all of GR, which is what Loop Quantum Gravity (LQG) is concerned with, and then to do a symmetry reduction (impose things as homogeneity, isotropy, etc.) and see what physics results from this. However, at present we do not have a quantized theory of gravity that is ready for use, so this is currently not an option. Therefore what one does in Loop Quantum Cosmology (LQC), is *first* to do the symmetry reduction – thereby obtaining classical cosmological models as e.g. FRW-models – and then to apply the ideas and techniques of LQG to those models in order to try to quantize them. Such models are much simpler than full GR, and therefore they are also much simpler to quantize. In this way LQC has already succeeded in quantizing a number of cosmological models successfully, thereby resolving the classical singularities. Although quantum theories of cosmology obtained by applying full LQG, whenever this will become possible, might give results that are different from those of LQC, the hope is that LQC really captures the main quantum effects affecting the scale factor.
We see something similar for the hydrogen atom. The hydrogen atom is a simple model, rather than a whole theory. This makes its quantization relatively easy. Usually the hydrogen atom is quantized by using ordinary quantum mechanics, and this works excellently well; all the main quantum effects in the system are incorporated by doing this. Ultimately, however, to quantize the hydrogen atom we should take full QED and apply it to the hydrogen atom. But most of the time we're not concerned with this, as the main quantum effects are the same for both approaches. We hope that the same will apply for LQC.

4.1 Classical Phase Space of Loop Quantum Gravity

4.1.1 Ashtekar-Barbero Variables

In this section I give a brief description of the Asktekar-Barbero formalism, on which LQG is based [1]. First we define new variables, the Asktekar-Barbero variables, that are related to the standard variables of GR. Then the constraints arising from the Hamiltonian formalism of GR are rewritten in terms of those variables. Since the Asktekar-Barbero variables are mathematically very nice objects, this makes it easier to do the quantization.

The Asktekar-Barbero variables are defined as follows. (Latin indices from the beginning of the alphabet $(a, b, \ldots)$ denote spatial indices whereas Latin indices from the middle of the alphabet $(i, j, \ldots)$ denote SU(2) indices that label new degrees of freedom introduced by the formalism.)
First we define the co-triad e_a^i, such that the induced spatial metric is given by $h_{ab} = e_a^i e_b^j \delta_{ij}$, where δ_j^i is the Kronecker delta. The triad e_i^a is then defined as its inverse: $e_i^a e_b^j = \delta_j^i \delta_b^a$. The configuration variable will be the Ashtekar-Barbero connection, given by

$$A_a^i = \Gamma_a^i + \gamma K_a^i \ , \tag{4.1}$$

where $0 \neq \gamma \in \mathbb{R}$ is known as the Immirzi parameter (whose value is fixed by calculations of black hole entropy), $K_a^i = K_{ab} e_j^b \delta^{ij}$ is the extrinsic curvature in triadic form, defined from the 'original' extrinsic curvature tensor K_{ab} as defined in (2.73) and Γ_a^i is the spin connection compatible with the densitized triad, i.e. it satisfies $D_b E_i^a + \epsilon_{ijk} \Gamma_b^j E^{ak} = 0$, where ϵ_{ijk} is the totally antisymmetric symbol and D_b is the spatial covariant derivative operator as defined in appendix A.

The conjugate momentum variable, corresponding to the Ashtekar-Barbero connection, is the densitized triad, given by

$$E_i^a = \sqrt{h}\, e_i^a \ , \tag{4.2}$$

where h is the determinant of the spatial metric. In these new variables the constraints read

$$\mathcal{C}^{grav} = \frac{1}{\sqrt{|det(E)|}}\epsilon_{ijk}\left[F_{ab}^i - (1+\gamma^2)\epsilon^i_{mn}K_a^m K_b^n\right]E^{aj}E^{bk} = 0 \ , \tag{4.3}$$

$$\mathcal{C}_a^{grav} = F_{ab}^i E_i^b = 0 \ , \tag{4.4}$$

$$G_i = \partial_a E_i^a + \epsilon_{ijk}\Gamma_a^j E^{ak} = 0 \ , \tag{4.5}$$

where $F_{ab}^i = \partial_a A_b^i - \partial_b A_a^i + \epsilon_{ijk}A_a^j A_b^k$ is the curvature tensor of the Ashtekar-Barbero connection. The first two of these equations are respectively the Hamiltonian constraint (2.77) and the diffeomorphism constraint (2.78), and the third equation is an additional constraint, called the gauge constraint or the Gauss constraint, that fixes the SU(2) freedom of the formalism. Furthermore we have the nonzero Poisson bracket

$$\{A_i^a(x), E_b^j(y)\} = 8\pi G\gamma\delta_b^a\delta_j^i\delta(x-y) \ . \tag{4.6}$$

4.1.2 Holonomy-Flux Algebra

In LQG the basic variables that will be quantized are holonomies around loops and fluxes through these loops. That explains the name Loop Quantum Gravity. The holonomy of the connection A along a curve e is defined by

$$h_e(A) = \mathcal{P}e^{\int_e dx^a A_a^i(x)\tau_i} \ . \tag{4.7}$$

Here $\mathcal{P}$ denotes the path ordering and τ_i are the generators of (a representation of) the $\mathfrak{su}(2)$ Lie algebra such that $[\tau_i, \tau_j] = \epsilon_{ij}^k\tau_k$. The holonomies contain the same information as the connection. Note that, since a holonomy is the exponentiation of an element of $\mathfrak{su}(2)$, it is an element of SU(2) (as will be shown in section 5 for the fundamental representation). The variables conjugate to the holonomies are the fluxes of E_i^a over surfaces S and smeared with an $\mathfrak{su}(2)$-valued function f^i:

$$E(S,f) = \int_S f^i E_i^a \epsilon_{abc} dx^b dx^c \ . \tag{4.8}$$

We then have the nonzero Poisson bracket

$$\{E(S,f), h_e(A)\} = 2\pi G\gamma\,\epsilon(e,S) f^i\tau_i h_e(A) \ , \tag{4.9}$$

where $\epsilon(e,S)$ represents the regularization of the Dirac delta: it vanishes if e does not intersect S or if $e \subset S$; and it is given by $\epsilon(e,S) = \pm 1$ whenever e and S intersect in one point. The sign then depends on the relative orientation between e and S.

4.2 The FRW-model ($k = 0$, $\Lambda = 0$) Coupled to a Scalar Field

The simplest (non-trivial) model in LQC is the flat FRW-model with a vanishing cosmological constant, coupled to a mass-less scalar field. This model is described classically in section 2.3.2. In LQC this model can be quantized completely (even analytically) and therefore it is sometimes called solvable Loop Quantum Cosmology (sLQC). In this case the diffeomorphism constraint (4.4) and the Gauss constraint (4.5) are automatically satisfied, and the integral version of the gravitational Hamiltonian constraint (4.3) (which is now precisely the gravitational Hamiltonian) is given by

$$C^{grav} = \int_\Sigma d^3x\, \mathcal{C}^{grav} = -\frac{1}{\gamma^2}\int_\Sigma d^3x\, \frac{\epsilon_{ijk}F^i_{ab}E^{aj}E^{bk}}{\sqrt{|det(E)|}} \ . \tag{4.10}$$

Integrals such as the above will generally diverge due to homogeneity. Therefore the analysis (and thereby the integrals) is usually restricted to a finite cell $\mathcal{V}$ of volume V. Since we are dealing with a homogeneous universe, the behavior of this cell will be representative of the whole universe.

The metric is given by the flat FRW-metric that can be written as $ds^2 = -dt^2 + h_{ab}dx^a dx^b$, where $h_{ab} = a(t)^2 \mathring{h}_{ab}$, for a *fiducial metric* $\mathring{h}_{ab}$ that does not depend on time (or position). We define also the fiducial triad $\mathring{e}^a_i$ and co-triad $\mathring{e}^i_a$ as the ones that correspond to the fiducial metric. (More generally, in the following, whenever there is a circle above a variable, it corresponds to the fiducial metric.) The volume of the cell $\mathcal{V}$ with respect to the fiducial metric we will denote by V_0. This implies that $V = V_0\, a^3$.
We can describe the Ashtekar-Barbero connection and the densitized triad by a single variable c and p, respectively,

$$A^i_a = c\,(V_0)^{-1/3}\ \mathring{e}^i_a, \quad E^a_i = p\,(V_0)^{-2/3}\sqrt{\mathring{h}}\,\mathring{e}^a_i \ , \tag{4.11}$$

where c and p are conjugate variables such that $\{c, p\} = \frac{8\pi G\gamma}{3}$ and p is related to the scale factor a by

$$a = V_0^{-1/3}\sqrt{|p|} \ , \tag{4.12}$$

This suggests that c and p are convenient variables in LQC, which is indeed the case. p is positive (negative) if the orientation of fiducial triad is equal (opposite) to the orientation to the physical triad. Note that $V = |p|^{3/2}$.

Holonomies and fluxes
Now we turn to the holonomies and fluxes. Again due to homogeneity, it suffices to consider only holonomies $h^\mu_i(c)$ along the (straight) edges of $\mathcal{V}$ with oriented length $\mu(V_0)^{1/3}$ for a constant parameter μ.[13] The holonomy in the direction i is given by

$$h^\mu_i(c) = e^{\mu c \tau_i} = \cos\left(\frac{\mu c}{2}\right)\mathbb{1} + 2\sin\left(\frac{\mu c}{2}\right)\tau_i \ . \tag{4.13}$$

For the fundamental representation of $\mathfrak{su}(2)$ the last equality in the above expression will be shown to be true in section (5.1). The configuration variables are then the matrix elements of the above expression; they are given by

$$\mathcal{N}_\mu(c) = e^{\frac{i\mu c}{2}} \ . \tag{4.14}$$

[13]These holonomies and fluxes form a complete set for describing the phase space.

As for the conjugate momenta, it is sufficient to consider only fluxes across the faces of $\mathcal{V}$. Those are given by

$$E(S,f) = p(V_o)^{-2/3} A_{S,f}, \tag{4.15}$$

where $A_{S,f}$ is the fiducial area of S times an orientation factor depending on f. So essentially the flux is described by a single variable p. The geometry part of the phase space is then described by the variables $\mathcal{N}_\mu(c)$ and p, whose Poisson bracket is given by

$$\{\mathcal{N}_\mu(c), p\} = i\frac{4\pi G\gamma}{3}\mu\mathcal{N}_\mu(c) \ , \tag{4.16}$$

and these are the basic variables that are quantized in sLQC.

Loop quantization

By quantizing this phase space, one obtains, instead of an operator representing c, an operator representing it's exponential, the holonomy $\mathcal{N}_\mu(c) = e^{\frac{i\mu c}{2}}$. Then the gravitational Hilbert space is not the standard Hilbert space (i.e. the space of square integrable functions), but rather it is $L^2(\mathbb{R}_{\text{Bohr}}, d\mu_{\text{Bohr}})$, i.e. the space of square integrable functions on the Bohr compactification of the real line [13]. It is, however, more convenient to work in the momentum representation. The quantum states $\mathcal{N}_\mu(c)$ can then be represented as kets $|\mu\rangle$ and these span the corresponding Hilbert space. An important feature is that these states are normalizable with respect to the discrete (!) inner product,

$$\langle\mathcal{N}_\mu(c)|\mathcal{N}_{\mu'}(c)\rangle = \delta_{\mu,\mu'} \ . \tag{4.17}$$

The basic operators then act on these states in the following way

$$\hat{\mathcal{N}}_{\mu'}|\mu\rangle = |\mu+\mu'\rangle \ , \tag{4.18}$$

$$\hat{p}|\mu\rangle = p(\mu)|\mu\rangle \ , \tag{4.19}$$

where $p(\mu) = \frac{4\pi G\gamma}{3}\mu$. The expression (4.18) follows immediately from the expression of $\mathcal{N}_\mu(c)$ and $|\mu\rangle$, and (4.19) then follows from the Dirac rule (3.1).

On the other hand, for the scalar field we one uses a standard representation, where $\hat{\phi}$ acts by multiplication, $\hat{\phi} = \phi$, and $\hat{p}_\phi$ acts by differentiation, $\hat{p}_\phi = -i\hbar\partial\phi$.

The Hamiltonian constraint

In terms of c and p the gravitational part of the (integrated[14]) Hamiltonian constraint (4.10) reads

$$C^{\text{grav}} = -\frac{3c^2\sqrt{|p|}}{8\pi G\gamma^2} \ . \tag{4.20}$$

The matter part of the Hamiltonian constraint is obtained from the Lagrangian density for a homogeneous mass-less scalar field: $\mathcal{L} = \frac{1}{2}\dot{\phi}^2$. Since the field is homogeneous, we can look directly at the Lagrangian, which is simply given by $L = V\mathcal{L} = \frac{1}{2}\dot{\phi}^2 V$. The momentum conjugate to ϕ is then

[14] From now on we will only work with the integrated constraint, so I will drop the word integrated from now on.

given by $p_\phi = \partial L/\partial\dot{\phi} = \dot{\phi}V$, so that the matter Hamiltonian is given by $C = p_\phi\dot{\phi} - L = \frac{1}{2}\dot{\phi}^2 V$. As a function of ϕ and p_ϕ the matter Hamiltonian reads

$$C^{\text{matt}} = \frac{p_\phi^2}{2V} = \frac{p_\phi^2}{2|p|^{3/2}} \ . \tag{4.21}$$

Adding the matter Hamiltonian to the gravitational constraint, as explained in section 2.4.2, we obtain the total Hamiltonian constraint

$$C = C^{\text{matt}} + C^{\text{grav}} = \frac{p_\phi^2}{2|p|^{3/2}} - \frac{3c^2\sqrt{|p|}}{8\pi G\gamma^2} = 0 \ . \tag{4.22}$$

For our present purposes, however, it is more convenient to use an even different pair of gravitational variables: b and v, related by the previous variables by

$$b = \frac{c}{\gamma\sqrt{|p|}}, \quad v = \frac{|p|^{3/2}\text{sgn}(p)}{4\pi G} \ . \tag{4.23}$$

In these variables we have $\{b, v\} = 1$ and the constraint reads

$$C = \frac{p_\phi^2}{8\pi G|v|} - \frac{3}{2}b^2|v| = 0 \ . \tag{4.24}$$

The volume of the cell $\mathcal{V}$ is then given by $V = 4\pi G|v|$ and thus the behavior of the volume of this cell and, due to homogeneity, the volume of the entire universe is represented by the variable v.

The scalar field ϕ as internal time

In the quantized model, the scalar field ϕ is used as internal time, i.e. the quantum states evolve as functions of ϕ rather than t. This is because the generator of time evolution (i.e. the Hamiltonian) is identical to zero, and therefore it does not generate any evolution. I would like to illustrate this fact by considering the Schrödinger equation,

$$i\frac{\partial}{\partial t}\Psi = \hat{H}\Psi \ , \tag{4.25}$$

for a wavefunction Ψ and Hamiltonian operator $\hat{H}$. If the Hamiltonian vanishes, then the corresponding Hamiltonian operator must analogously act as the zero-operator on all quantum states, and hence one is left with the simple fact that

$$\frac{\partial}{\partial t}\Psi = 0 \ , \tag{4.26}$$

so there is no time evolution. Intuitively we can see the problem by realizing that time is not uniquely defined in GR; different observers can measure different times between the same events. This realization suggests that the evolution of the quantum states should not be described by a particular time coordinate, but rather it should be described in an observer independent way. A way to achieve this is to regard ϕ as internal time. Then, in LQC, one can obtain wave functions $\Psi(v, \phi)$, and these are in fact independent of the observer.

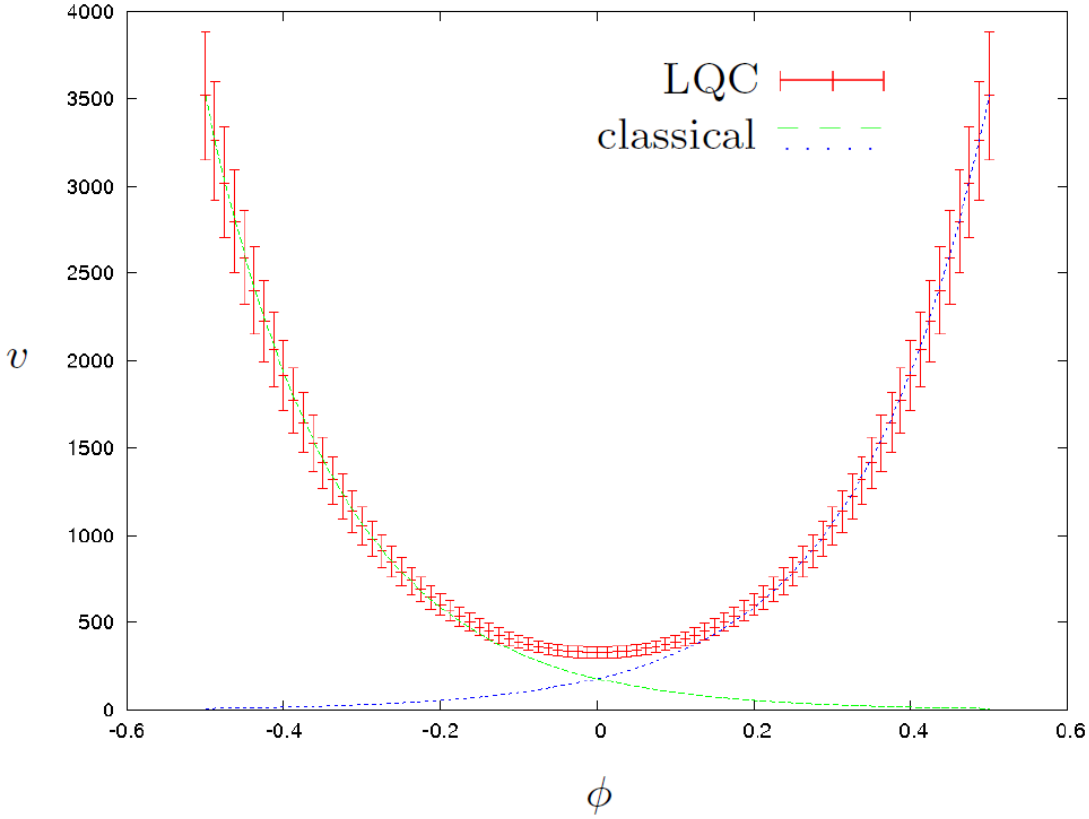


Figure 1: In the red, we see the evolution of the expectation value of the quantum operator in LQC, corresponding to the classical variable v , which is proportional to the volume of cell $\mathcal{V}$ and due to homogeneity represents the evolution of the whole universe. The blue and green curves show the classical trajectories of v. The first ones to produce plots like this were Ashtekar A., Pawlowski T. and Singh P. This plot is taken from [14].

With numerical simulations it has been shown that for semi-classical states[15] the expectation value of the volume operator (corresponding to the cell $\mathcal{V}$) undergoes a quantum bounce as the volume reaches a minimum value. This is illustrated in figure 1. Also, by restricting the analysis to a superselection sector (which is, informally, a subspace of the full kinimatical Hilbert space that is closed under the action of the relevant operators on the Hilbert space), the dynamics can be computed analytically. The expectation value of $\hat{V}$ (again for semi-classical states) then satisfies

$$\langle|\hat{V}|\rangle = V_+ e^{\sqrt{12\pi G}\phi} + V_- e^{-\sqrt{12\pi G}\phi} \ , \tag{4.27}$$

where V_+ and V_- are positive constants. The expectation value never reaches zero and the singularities are thus absent and replaced by a bounce. This shows that the bounce is a generic feature of

[15] Semi-classical states are states for which the wave functions are highly peaked around the classical trajectory.

sLQC.

4.2.1 Effective Dynamics

For semi-classical states the leading quantum corrections on the dynamics of the model can be formulated in a classical way by something that is known as effective dynamics. Basically all one has to do is make the replacement ('regularization') $b \to \sin(\lambda b)/\lambda$ for a constant $\lambda \in \mathbb{R}$. The constraint (4.24) then reads

$$C_{(\mathrm{eff})} = -\frac{3}{2}\frac{\sin^2(\lambda b)}{\lambda^2}|v| + \frac{p_\phi^2}{8\pi G|v|} = 0 \ , \tag{4.28}$$

and, for semi-classical states, the classical dynamics generated by this constraint will resemble the dynamics of the expectation values of the quantum operators extremely well. Therefore the effective dynamics serves as a very useful tool for various purposes. In the following, we will derive the dynamical evolution of b and v resulting from the effective dynamics by solving the Hamilton equations in a straightforward way. Later (section 6) we will also derive the same dynamical evolution by using group theoretical arguments, and for this we will use the concepts that will first be developed in section 5.

Now, to compute the dynamics generated by the effective constraint, we change variables once again, according to the canonical transformation $v \to v' = v/\lambda$, $b \to b' = \lambda b$, and redefine $v := v'$, $b := b'$, the constraint then reads

$$C_{(\mathrm{eff})} = -\frac{3}{2}\frac{\sin^2(b)}{|\lambda|}|v| + \frac{p_\phi^2}{8\pi G|\lambda||v|} = 0 \ . \tag{4.29}$$

We will use ϕ as internal time, as is done in the genuine quantum theory, and so compute the evolution of the system as it changes with ϕ. We do this as follows. First we write the Hamilton equation

$$\frac{\partial \phi}{\partial t} = \{\phi, C\} = \frac{\partial C}{\partial p_\phi} \tag{4.30}$$

in a convenient way:

$$\frac{\partial t}{\partial \phi} = \frac{\partial p_\phi}{\partial C} \ . \tag{4.31}$$

The other two Hamilton equations that we will use are

$$\frac{\partial b}{\partial t} = \{b, C\} = \frac{\partial C}{\partial v} = \frac{\partial C}{\partial p_\phi}\frac{\partial p_\phi}{\partial v} \ , \tag{4.32}$$

$$\frac{\partial v}{\partial t} = \{v, C\} = -\frac{\partial C}{\partial b} = -\frac{\partial C}{\partial p_\phi}\frac{\partial p_\phi}{\partial b} \ . \tag{4.33}$$

$$\tag{4.34}$$

Now we write out the derivatives of b and v with respect to ϕ using the chain rule, and substitute (4.31), (4.32) and (4.33) to obtain what we will call the *internal Hamilton equations.*

$$\frac{\partial b}{\partial \phi} = \frac{\partial b}{\partial t}\frac{\partial t}{\partial \phi} = \left(\frac{\partial C}{\partial p_\phi}\frac{\partial p_\phi}{\partial v}\right)\left(\frac{\partial p_\phi}{\partial C}\right) = \frac{\partial p_\phi}{\partial v} \tag{4.35}$$

$$\frac{\partial v}{\partial \phi} = \frac{\partial v}{\partial t}\frac{\partial t}{\partial \phi} = \left(-\frac{\partial C}{\partial p_\phi}\frac{\partial p_\phi}{\partial b}\right)\left(\frac{\partial p_\phi}{\partial C}\right) = -\frac{\partial p_\phi}{\partial b}\ . \tag{4.36}$$

From these equations we see that we can regard p_ϕ as the *internal Hamiltonian* generating evolution in ϕ. The phase space is then reduced to the two dimensions constituted by b and v and the Hamiltonian constraint (4.29) can be rewritten to write the internal Hamiltonian as a function of those variables,

$$H_{\text{int}} = p_\phi = \pm\sqrt{12\pi G}v\sin(b)\ . \tag{4.37}$$

Evolution in ϕ is then given simply by the internal Hamilton equations

$$\partial_\phi v = \{v, H_{\text{int}}\}, \quad \partial_\phi b = \{b, H_{\text{int}}\}\ . \tag{4.38}$$

Noting the constant factor in the internal Hamiltonian, these equations take a simple form for the internal time variable $\tau := \sqrt{12\pi G}\,\phi$, namely

$$\partial_\tau v = \cos(b)v, \quad \partial_\tau b = -\sin(b) \tag{4.39}$$

The solutions are given by

$$v(\tau) = v_0\cosh(\tau - \tau_0), \quad b(\tau) = \arccos(\tanh(\tau - \tau_0))\ . \tag{4.40}$$

Since $|v|$ is proportional to the volume, the volume goes as $V \propto \cosh(\tau - \tau_0)$ which coincides with expectation value of the volume operator in the analytic solutions of the quantum model (4.27) if $V_+ \approx V_-$.

5 Groups, Algebras and Deformations

In this section we will develop the mathematical concepts needed 1) to identify the structure of the phase space of the effective dynamics, 2) to use this structure to compute the evolution of the variables b and v, and 3) to deform this structure and thereby deform the (internal) Hamiltonian and the dynamics.

Definition 1. *A group is a set G equipped with a binary operation $G \times G \to G : (g_1, g_2) \mapsto g_1 * g_2$ that satisfies the following conditions:*

- *Associativity: for all $g_1, g_2, g_3 \in G : g_1 * (g_2 * g_3) = (g_1 * g_2) * g_3$*
- *There exists a neutral element $e \in G$ such that for all $g \in G : e * g = g * e = g$*
- *for all $g \in G$ there exists an inverse $g^{-1} \in G$ such that $g * g^{-1} = g^{-1} * g = e$.*

In case of groups where the elements are matrices, it is assumed throughout this text that the binary operation is matrix multiplication. These groups are called matrix groups.

Definition 2. *A Lie group L is a group which is at the same time a smooth (finite-dimensional) manifold such that the binary operation $G \times G \to G : (g_1, g_2) \mapsto g_1 * g_2$ and the inverse $G \times G \to G : g \mapsto g^{-1}$ are smooth.*

Definition 3. *A Lie algebra $(V, [\,,\,])$ over $\mathbb{K}$, where $\mathbb{K}$ can be $\mathbb{R}$ or $\mathbb{C}$, is a vector space V over $\mathbb{K}$ together with a binary operation $V \times V \to V : (v_1, v_2) \mapsto [v_1, v_2]$ (the Lie bracket) such that for all $X, Y, Z \in V$:*

- $[X, Y] = -[Y, X]$ *(antisymmetry)*
- $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ *and* $[X, aY + bZ] = a[X, Y] + b[X, Z]$ *for all* $a, b \in \mathbb{K}$ *(bilinearity)*
- $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ *(Jacobi identity).*

The basis vectors of V are called the generators of the Lie algebra.

For any Lie group there is a corresponding Lie algebra, which is the tangent space of the Lie group at the identity. Conversely, for any (finite-dimensional) Lie algebra there is a corresponding (but not always unique) Lie group. Elements of a Lie algebra can be viewed as infinitesimal versions of elements of the corresponding Lie group. This will be demonstrated for the Lie groups SU(2) and SU(1,1) in section 5.1 and 5.2, respectively.

Definition 4. *A $\mathbb{K}$-algebra $(V, \cdot)$ (also called an algebra over $\mathbb{K}$) is a vector space V over a field $\mathbb{K}$ equipped with a bilinear product $V \times V \to V : (v_1, v_2) \mapsto v_1 \cdot v_2$ such that for all $x, y, z \in V$:*

- $(x + y) \cdot z = x \cdot z + y \cdot z$
- $x \cdot (y + z) = x \cdot y + x \cdot z$
- $(ax) \cdot (by) = (ab)(x \cdot y)$ *for all* $a, b \in \mathbb{K}$.

An associative $\mathbb{K}$-algebra is then a $\mathbb{K}$-algebra that is associative, i.e.

- $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ *for all* $x, y, z \in V$

Definition 5. *A Poisson algebra* $(V, \cdot, [\,,\,])$ *over* $\mathbb{K}$*, where* $\mathbb{K}$ *can be* $\mathbb{R}$ *or* $\mathbb{C}$*, is a vector space* V *together with two binary operations* $V \times V \to V : (v_1, v_2) \mapsto v_1 \cdot v_2$ *(multiplication) and* $V \times V \to V : (v_1, v_2) \mapsto [v_1, v_2]$ *(the Lie bracket) such that*

- $(V, [\,,\,])$ *is a Lie algebra over* $\mathbb{K}$
- $(V, \cdot)$ *is an associative* $\mathbb{K}$*-algebra*
- *the Lie bracket satisfies the Leibniz rule* $[X \cdot Y, Z] = X \cdot [Y, Z] + [X, Z] \cdot Y$ *for all* $X, Y, Z \in V$*.*

Since the Poisson bracket satisfies antisymmetry, bilinearity and the Jacobi identity (see appendix B), it can often be considered a Lie bracket. And since it also satisfies the Leibniz rule, an associative $\mathbb{K}$-algebra equipped with the Poisson bracket (a phase space) can often be considered a Poisson algebra. Throughout this thesis, whenever I refer to a Poisson algebra, it is implied that the Lie brackets are given by Poisson brackets.

5.1 The Lie Group SU(2) and its Lie Algebra $\mathfrak{su}(2)$

Although the Lie group SU(2) and its Lie algebra $\mathfrak{su}(2)$ are not made use of in this thesis, apart from mentioning them in the description of LQG, a brief description of them is will be given in this section. This is because they are well known and they have a lot in common with SU(1,1) and $\mathfrak{su}$(1,1). If the reader is familiar with SU(2), he/she may find it beneficial to relate the concepts that will be developed for SU(1,1) to SU(2). Also, the deformations of $\mathfrak{su}(2)$, that are also relatively well known, have played an important role in developing the deformations of $\mathfrak{su}$(1,1) in this thesis.

The Lie group SU(2) is the group consisting of all complex 2×2 unitary matrices with determinant equal to 1. A unitary matrix is a matrix U who's Hermitian conjugate is equal to its inverse, i.e. $U^\dagger = U^{-1}$. This implies that $U^\dagger U = \mathbb{1}$. We can thus reformulate: SU(2) is the group consisting of complex 2x2 matrices for which $det(U) = 1$ and

$$U^\dagger \epsilon U = \epsilon \ , \tag{5.1}$$

where $\epsilon = \mathbb{1}$. Now it might seem strange that we have introduced ϵ here, but in this way we will be able to see a clear correspondence between the groups SU(2) and SU(1,1), since the only difference between them is ϵ. This matrix ϵ is actually a metric that defines a scalar product $\langle z|\epsilon|w\rangle$ of two vectors $|z\rangle$ and $|w\rangle$ living in the vector space that U acts on and their Hermitian conjugates $\langle z|$ and $\langle w|$. And whatever the exact form of ϵ may be, a matrix that satisfies (5.1) leaves this scalar product invariant, for we have

$$\langle z'|\epsilon|w'\rangle = \langle z|U^\dagger \epsilon U|w\rangle = \langle z|\epsilon|w\rangle \ . \tag{5.2}$$

For SU(2) this is the conventional scalar product and the elements of SU(2) therefore leave angles and distances invariant. Explicitly, the group elements $U \in$ SU(2) are given by

$$U \in \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : \quad \alpha, \beta \in \mathbb{C}, \quad |\alpha|^2 + |\beta|^2 = 1 \right\} , \tag{5.3}$$

where a bar denotes complex conjugation.

The Lie algebra $\mathfrak{su}(2)$ is the vector space V spanned by the matrices

$$\tau_1 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}; \quad \tau_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \quad \tau_3 = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \tag{5.4}$$

equipped with the Lie brackets (which just the commutators):

$$[\tau_1, \tau_2] = \tau_3, \quad [\tau_2, \tau_3] = \tau_1, \quad [\tau_3, \tau_1] = \tau_2, \tag{5.5}$$

which in some texts is written very compactly, adopting the summation convention, as

$$[\tau_i, \tau_j] = \epsilon_{ij}{}^{k}\tau_k, \tag{5.6}$$

where $i, j, k = 1, 2, 3$. If, however, the totally antisymmetric symbol, $\epsilon_{ij}{}^{k}$, needs to specified explicitly, this is probably not worth the effort, for it is given by

$$\epsilon_{ij}{}^{k} = \begin{cases} 1 \text{ if ijk is an even permutation of 123} \\ -1 \text{ if ijk is an odd permutation of 123} \\ \qquad\qquad 0 \text{ otherwise} \end{cases}, \tag{5.7}$$

which is quite a big chunk of formula.

Any element of SU(2) can be obtained by exponentiating an element of the $\mathfrak{su}(2)$ algebra. To see this, we write a general $\mathfrak{su}(2)$ element $\mathfrak{g}$ as a linear combination of its basis elements, $\mathfrak{g} = u_1\tau_1 + u_2\tau_2 + u_3\tau_3 \equiv \vec{u}\cdot\vec{\tau}$, with $\vec{u} \in \mathbb{R}^3$. Then, using $(\vec{u}\cdot\vec{\tau})^2 = -|u|^2\,\mathbb{1}$ and substituting this in the Taylor expansion of the exponential function, we find that

$$\begin{aligned} e^{\mathfrak{g}} = e^{\vec{u}\cdot\vec{\tau}} &= \cos\left(\frac{|u|}{2}\right)\mathbb{1} + 2\sin\left(\frac{|u|}{2}\right)\frac{\vec{u}\cdot\vec{\tau}}{|u|} \\ &= \begin{pmatrix} \cos\left(\frac{|u|}{2}\right) + i\frac{u_3}{|u|}\sin\left(\frac{|u|}{2}\right) & -\frac{u_2}{|u|}\sin\left(\frac{|u|}{2}\right) + i\frac{u_1}{|u|}\sin\left(\frac{|u|}{2}\right) \\ \frac{u_2}{|u|}\sin\left(\frac{|u|}{2}\right) + i\frac{u_1}{|u|}\sin\left(\frac{|u|}{2}\right) & \cos\left(\frac{|u|}{2}\right) - i\frac{u_3}{|u|}\sin\left(\frac{|u|}{2}\right) \end{pmatrix} \in \mathrm{SU}(2)\ . \end{aligned} \tag{5.8}$$

One can check that this matrix is indeed of the form (5.3), which proves that it is an element of SU(2). From (5.3) we also note that SU(2) has three independent real parameters[16]. In (5.8) the parameterized elements of SU(2) also have three independent parameters. This proves that all elements of SU(2) can be obtained by this parameterization and therefore all elements of SU(2) can be obtained by exponentiating the elements of its Lie algebra. Conversely, from this parameterization, we easily see that when we take u to be infinitesimal, and thus only consider the 0^{th} and 1^{st} order terms in the Taylor expansion, that

$$g = \mathbb{1} + \vec{u}\cdot\vec{\tau}\ , \tag{5.9}$$

since $\sin(x) = x + \mathcal{O}(x^3)$ and $\cos(x) = 1 + \mathcal{O}(x^2)$. This shows that $\vec{u}\cdot\vec{\tau}$ is the tangent vector at the identity. The tangent space is thus the space spanned by the generators τ_i, which is precisely

[16]For there are two complex variables, α and β, that constitute four real parameters, and we have one (real) equation for them that takes away one parameter.

the definition we had adopted for the $\mathfrak{su}(2)$ Lie algebra. We can thus conclude that indeed $\mathfrak{su}(2)$ is the tangent space of SU(2) at the identity, as was mentioned earlier.

Often, the notion '$\mathfrak{su}(2)$ algebra' is used to denote a Lie algebra with Lie brackets (5.6) in a more general sense; the Lie bracket is not necessarily the commutator and the underlying vector space can be any vector space V. If there exists a basis of vectors $\tilde{\tau}_i$ for V that satisfies $[\tilde{\tau}_i, \tilde{\tau}_j] = \epsilon_{ij}{}^k \tilde{\tau}_k, \quad i, j, k \in \{1, 2, 3\}$, where $[\,,\,]$ is a general Lie bracket, as according to definition 4, then we say that V equipped with $[\,,\,]$ is an $\mathfrak{su}(2)$ algebra, or that the *generators* $\tilde{\tau}_i$ (or any other complete set of basis vectors of V) close an $\mathfrak{su}(2)$ algebra.

5.2 The Lie Group SU(1,1) and its Lie Algebra $\mathfrak{su}$(1,1)

The Lie group SU(1,1) is the group consisting of all complex 2×2 matrices U for which $\det(U) = 1$ and

$$U^\dagger \epsilon U = \epsilon \ , \quad \text{where} \quad \epsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{5.10}$$

Note that the definition is the same as for SU(2), except for the fact that ϵ has changed. So therefore SU(1,1) also has the property that it leaves the scalar products invariant (see previous section), yet in this case the scalar product is defined by the metric $diag(1, -1)$. Explicitly, the group elements $U \in$ SU(1,1) are given by

$$U \in \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} : \quad \alpha, \beta \in \mathbb{C}, \quad |\alpha|^2 - |\beta|^2 = 1 \right\} . \tag{5.11}$$

The Lie algebra $\mathfrak{su}$(1,1) is obtained by considering the vector space V spanned by the matrices

$$\sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad \sigma_2 = -\frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}; \quad \sigma_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.12}$$

equipped with the Lie brackets given by the commutators:

$$[\sigma_3, \sigma_1] = i\sigma_2; \quad [\sigma_3, \sigma_2] = -i\sigma_1; \quad [\sigma_1, \sigma_2] = -i\sigma_3 \ . \tag{5.13}$$

The group SU(1,1) can be obtained from its Lie algebra by exponentiation, just like was the case for SU(2). But it is a bit more tricky this time. We again write a general element of $\mathfrak{su}$(1,1) as a linear combination of the generators, $\mathfrak{g} = \vec{u} \cdot \vec{\sigma}$, where we consider $\vec{u}$ to live in $\mathbb{R}^3$ with metric $diag(-1, -1, +1)$. Then its norm is given by $|\vec{u}|^2 = -u_1^2 - u_2^2 + u_3^2$ and we distinguish three cases: (i) vanishing norm, (ii) positive norm and (iii) negative norm. By realizing that

$$(2\vec{u} \cdot \vec{\sigma})^2 = \begin{pmatrix} |u|^2 & 0 \\ 0 & |u|^2 \end{pmatrix} \ \Rightarrow \ \begin{cases} \boldsymbol{(i)} & (2\vec{u} \cdot \vec{\sigma})^2 = 0 \\ \boldsymbol{(ii)} & (2\vec{u} \cdot \vec{\sigma})^2 = \text{abs}\left(|\vec{u}|^2\right) \mathbb{1} \\ \boldsymbol{(iii)} & (2\vec{u} \cdot \vec{\sigma})^2 = -\text{abs}\left(|\vec{u}|^2\right) \mathbb{1} \end{cases} \tag{5.14}$$

and substituting this in the Taylor expansion of the exponential, we obtain

$$\begin{aligned}
&\textbf{\textit{(i)}} \quad e^{i\vec{u}\cdot\vec{\sigma}} = \mathbb{1} + i\vec{u}\cdot\vec{\sigma} && \in \mathrm{SU}(1,1) \\
&\textbf{\textit{(ii)}} \quad e^{i\vec{u}\cdot\vec{\sigma}} = \cos\left(\frac{-|\vec{u}|^2}{2}\right)\mathbb{1} + 2i\sin\left(\frac{|\vec{u}|}{2}\right)\frac{\vec{u}\cdot\vec{\sigma}}{|\vec{u}|} && \in \mathrm{SU}(1,1) \\
&\textbf{\textit{(iii)}} \quad e^{i\vec{u}\cdot\vec{\sigma}} = \cosh\left(\frac{\sqrt{-|\vec{u}|^2}}{2}\right)\mathbb{1} + 2i\sinh\left(\frac{\sqrt{-|\vec{u}|^2}}{2}\right)\frac{\vec{u}\cdot\vec{\sigma}}{\sqrt{-|\vec{u}|^2}} && \in \mathrm{SU}(1,1)
\end{aligned} \tag{5.15}$$

and these are all elements of SU(1,1), since they are of the form (5.11). And because we have the elements parameterized by the three independent components of $\vec{u}$, and SU(1,1) has three independent parameters as well, we can obtain all elements of SU(1,1) by exponentiating elements of $\mathfrak{su}$(1,1). Conversely, by looking at parameterized group elements we see that the tangent space of SU(1,1) at the identity is indeed precisely the $\mathfrak{su}$(1,1) algebra.

Just as is the case with $\mathfrak{su}(2)$, often the notion '$\mathfrak{su}$(1,1) algebra' is used to denote a Lie algebra with Lie brackets (5.6) in a more general sense; the Lie bracket is not necessarily given by the commutator and the underlying vector space can be any vector space V. If there exists a basis of vectors $\tilde{\sigma}_i$ for V, for $i = 1, 2, 3$, that satisfy

$$[\tilde{\sigma}_3, \tilde{\sigma}_1] = i\tilde{\sigma}_2; \quad [\tilde{\sigma}_3, \tilde{\sigma}_2] = -i\tilde{\sigma}_1; \quad [\tilde{\sigma}_1, \tilde{\sigma}_2] = -i\tilde{\sigma}_3 \ , \tag{5.16}$$

where $[\, ,]$ is a general Lie bracket, as according to definition 4, then we say that V equipped with $[\, ,]$ is an $\mathfrak{su}$(1,1) algebra, or that the *generators* $\tilde{\sigma}_i$ (or any other complete set of basis vectors of V) close an $\mathfrak{su}$(1,1) algebra.
In view of later application, a useful set of basis vectors is the set of $J_+ = \tilde{\sigma}_1 + i\tilde{\sigma}_2$, $J_- = \tilde{\sigma}_1 - i\tilde{\sigma}_2$ and $J_3 = \tilde{\sigma}_3$, for which the Lie brackets become

$$[J_3, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = -2J_3 \ . \tag{5.17}$$

Another useful set of generators is just the set of J_i, multiplied by $(-i) : K_+ = -iJ_+,\ K_- = -iJ_-$ and $K_3 = -iJ_3$. These have the Lie brackets

$$[K_3, K_\pm] = \mp iJ_\pm, \quad [K_+, K_-] = 2iK_3 \ . \tag{5.18}$$

5.2.1 Realization of $\mathfrak{su}$(1,1)

We define two complex spinor variables z_1, z_2 (it will become clear later why we call them spinor variables), with canonical Poisson brackets

$$\{z_i, \bar{z_j}\} = -i\delta_{ij}, \quad \{z_1, z_2\} = 0 \ , \tag{5.19}$$

where a bar denotes complex conjugation. $\mathfrak{su}$(1,1) can be realized as a Poisson algebra by the generators

$$K_+ = \bar{z_1}\bar{z_2}, \quad K_- = z_1 z_2 \quad K_3 = \tfrac{1}{2}(z_1\bar{z_1} + z_2\bar{z_2}) \ . \tag{5.20}$$

These satisfy

$$\{K_3, K_\pm\} = \mp iK_\pm, \quad \{K_+, K_-\} = 2iK_3 \ . \tag{5.21}$$

This way of realizing the algebra is sometimes called the realization with two harmonic oscillators, referring to the Poisson brackets of harmonic oscillator variables. We also define the operators

$$P_+ = \{K_+, \star\}, \quad P_- = \{K_-, \star\}, \quad P_3 = \{K_3, \star\} \ . \tag{5.22}$$

In appendix B.1 we show that these close an $\mathfrak{su}(1{,}1)$ Lie algebra, with the Lie brackets given by commutators, i.e.

$$[P_3, P_\pm] = \mp i P_\pm, \quad [P_+, P_-] = 2iP_3 \ . \tag{5.23}$$

In view of later application, we now consider a spinor $|z\rangle$ and its conjugate $\langle z|$, defined as

$$|z\rangle = \begin{pmatrix} z_1 \\ \bar{z}_2 \end{pmatrix} \ , \quad \langle z| = (\bar{z}_1 z_2) \ . \tag{5.24}$$

We establish a change of basis of the algebras by considering the generators

$$K_1 = \tfrac{1}{2}(K_+ + K_-) \ , \quad K_2 = \tfrac{1}{2i}(K_+ - K_-) \ , \quad K_3 \ , \tag{5.25}$$
$$P_1 = \tfrac{1}{2}(P_+ + P_-) \quad , \quad P_2 = \tfrac{1}{2i}(P_+ - P_-) \quad , \quad P_3 \ \ , \tag{5.26}$$

and we look at their action on the spinors[17]:

$$P_j|z\rangle = \{K_j, \star\}\,|z\rangle = \{K_j, |z\rangle\}, \quad j = 1, 2, 3 \ . \tag{5.28}$$

One finds that

$$P_j|z\rangle = i\sigma_j|z\rangle \tag{5.29}$$

with the σ_j as defined in (5.12). Exponentiating this, one finds that

$$e^{\vec{u}\cdot\vec{P}}|z\rangle = e^{\{\vec{u}\cdot\vec{K},\star\}}|z\rangle = U|z\rangle, \quad U = e^{i\vec{u}\cdot\vec{\sigma}} \in \mathrm{SU}(1{,}1), \tag{5.30}$$

where we have introduced the vectors $\vec{K} = (K_1, K_2, K_3)$, $\vec{P} = (P_1, P_2, P_3)$, $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ for the sake of simple notation. This means that the spinor $|z\rangle$ belongs to the fundamental representation of SU(1,1), which is in fact the reason that we call it a spinor.

5.3 The q-Deformation of $\mathfrak{su}(1{,}1)$ and its Realization

5.3.1 Definition of $\mathfrak{su}_q(1,1)$

The q-deformed $\mathfrak{su}(1{,}1)$ algebra, denoted as $\mathfrak{su}_q(1,1)$, is defined, in most references (e.g. [15]), by the Lie brackets

$$[K_3, K_\pm] = \pm K_\pm \ , \quad [K_+, K_-] = -[2K_3]_q \ , \tag{5.31}$$

[17] A different way of looking at this would be to simply define the action of the generators K_i on the spinors to be

$$K_i \triangleright |z\rangle \equiv \{K_i, |z\rangle\} \ , \quad i = 1, 2, 3 \ . \tag{5.27}$$

where we have defined the quantum number

$$[x]_q \equiv \frac{q^x - q^{-x}}{q - q^{-1}} = \frac{\sinh(\gamma x)}{\sinh(\gamma)}, \quad \gamma \equiv ln(q) \ , \tag{5.32}$$

in terms of the deformation parameter $q \in \mathbb{R}$.[18] For our present purposes it is, however, more convenient to use a slightly different definition. This is for the following reason.
In the Poisson algebra that we will identify on the sLQC phase space (in section 6), the Lie brackets are given by Poisson brackets. Upon quantizing the model those Poisson brackets will be replaced by commutators, following the canonical quantization procedure as described in section 3.1,

$$\{A, B\} \mapsto -i[\hat{A}, \hat{B}] \ . \tag{5.33}$$

At this point there is no reason not to be ambitious. Therefore we keep in mind that if we are to obtain a physically interesting system after applying the deformation procedure to our LQC model, probably we would like to quantize it. And then we would like the quantized system to satisfy the Lie brackets (given by the commutators) (5.31) of the $\mathfrak{su}_q(1,1)$ algebra. This implies that we should have, on the classical phase space, Poisson brackets that satisfy

$$\{K_3, K_\pm\} = \mp i K_\pm, \quad \{K_+, K_-\} = i[2K_3]_q \ . \tag{5.34}$$

This is what we refer to as the $\mathfrak{su}_q(1,1)$ algebra, throughout the rest of this thesis. This is not uncommon (see e.g. [16]). Note that in the limit $q \to 1$ ($\gamma \to 0$) (or by explicitly setting $q = 1$) we recover the $\mathfrak{su}(1,1)$ algebra (5.18).

5.3.2 Realization of $\mathfrak{su}_q(1,1)$

Guided by the realization of $\mathfrak{su}_q(2)$ (the q-deformed $\mathfrak{su}(2)$ algebra) that is discussed in [16] we define the q-deformed spinor variables

$$w_1 = (\gamma \sinh\gamma)^{-1/4} \sqrt{\frac{\sinh(\gamma z_1 \bar{z_1})}{z_1 \bar{z_1}}} z_1 e^{i\gamma\alpha_1(z_1\bar{z_1})} \ , \tag{5.35}$$

$$w_2 = (\gamma \sinh\gamma)^{-1/4} \sqrt{\frac{\sinh(\gamma z_2 \bar{z_2})}{z_2 \bar{z_2}}} z_2 e^{i\gamma\alpha_2(z_2\bar{z_2})} \ , \tag{5.36}$$

in terms of the original spinor variables z_1 and z_2. Here $\alpha_1(z_1\bar{z_1})$ and $\alpha_2(z_2\bar{z_2})$ can be arbitrary real functions. Then $\mathfrak{su}_q(1,1)$ can be realized as a Poisson algebra by considering the generators,

$$Q_+ = \bar{w_1}\bar{w_2} \ , \quad Q_- = w_1 w_2 \ , \quad Q_3 = \frac{1}{2}(z_1\bar{z_1} + z_2\bar{z_2}) \ . \tag{5.37}$$

These satisfy (5.34),

$$\{Q_3, Q_\pm\} = \mp i Q_\pm, \quad \{Q_+, Q_-\} = i[2Q_3]_q \ . \tag{5.38}$$

Thus, we have established a general prodecure for deforming an $\mathfrak{su}$(1,1) Poisson algebra:

[18] Note that in the limit $q \to 1$ ($\gamma \to 0$) (or by explicitly setting $q = 1$) we recover the $\mathfrak{su}(1,1)$ algebra (5.17).

1. Identify the $\mathfrak{su}$(1,1) generators that satisfy (5.21)
2. Write the generators in terms of the spinor variables, in the form (5.20)
3. In the expressions for K_+ and K_-, replace the spinor variables with the corresponding q-deformed spinor variables: $z_i \to w_i$. Leave K_3 unchanged.

If these steps are followed, then the new generators should form an $\mathfrak{su}_q(1,1)$ Poisson algebra. Note that the procedure is only relevant if step two is possible, which is, of course, not always the case.

6 Group Theoretical Analysis of the Effective Dynamics

In this section we show how we can obtain the functions $v(\tau)$ and $b(\tau)$ also by using group theoretical considerations. We start from the phase space variables b and v satisfying the canonical Poisson bracket

$$\{b, v\} = 1 \tag{6.1}$$

and identify the following phase space functions

$$K_+ = ve^{ib} \ , \quad K_- = ve^{-ib} \ , \quad K_3 = v \ . \tag{6.2}$$

One can check easily that these form an $\mathfrak{su}(1,1)$ Poisson algebra, i.e. they satisfy

$$\{K_3, K_\pm\} = \mp i K_\pm \ , \quad \{K_+, K_-\} = 2iK_3 \ . \tag{6.3}$$

Changing the basis of the Poisson algebra by considering the real generators

$$K_1 = \tfrac{1}{2}(K_+ + K_-) = v\cos(b) \ , \quad K_2 = \frac{1}{2i}(K_+ - K_-) = v\sin(b), \quad K_3 = v \ , \tag{6.4}$$

we see that the internal[19] Hamiltonian $H_{\text{int}} = \pm\sqrt{12\pi G}v\sin(b)$ of the system is precisely the K_2 generator of the algebra times a constant factor,

$$H_{\text{int}} = \pm\sqrt{12\pi G}\, K_2 \ . \tag{6.5}$$

In the following we show that this implies that evolution in ϕ is given by an SU(1,1) transformation.

Wee will consider the minus sign in the definition of the Hamiltonian from this point on. Picking the other sign will only reverse the direction of (internal) time. Since K_+, K_-, K_3 satisfy the $\mathfrak{su}(1,1)$ relations with their Poisson brackets, following section 5.2.1, we know that the operators

$$P_+ = \{K_+, \star\} \ , \quad P_- = \{K_-, \star\} \ , \quad P_3 = \{K_3, \star\} \tag{6.6}$$

form an $\mathfrak{su}(1,1)$ Lie algebra with their commutators, i.e.

$$[P_3, P_\pm] = \mp i P_\pm \ , \quad [P_+, P_-] = 2iP_3 \ . \tag{6.7}$$

For this Lie algebra we do an analogous change in basis and we consider the generators

$$P_1 = \{K_1, \star\} \ , \quad P_2 = \{K_2, \star\} \ , \quad P_3 \ . \tag{6.8}$$

Now we look at the evolution of b and v. This is given by Hamilton's equations (4.38) and infinitesimally those can be written as

$$v(\phi) \to v(\phi+\delta\phi) = v(\phi) + \partial_\phi v\, \delta\phi = v(\phi) + \{v(\phi), H_{\text{int}}\}\, \delta\phi = v(\phi) - \{H_{\text{int}}, v(\phi)\}\, \delta\phi \ , \tag{6.9}$$

$$b(\phi) \to b(\phi+\delta\phi) = b(\phi) + \partial_\phi b\, \delta\phi = b(\phi) + \{b(\phi), H_{\text{int}}\}\, \delta\phi = b(\phi) - \{H_{\text{int}}, b(\phi)\}\, \delta\phi \ . \tag{6.10}$$

[19] I will drop the word 'internal' from now on, and just call the effective Hamiltonian the Hamiltonian.

In terms of operators acting on the functions b and v these transformations read

$$v \xrightarrow[\phi\to\phi+\delta\phi]{} (1-\delta\phi\{H_{\text{int}},\star\})\, v \ , \tag{6.11}$$

$$b \xrightarrow[\phi\to\phi+\delta\phi]{} (1-\delta\phi\{H_{\text{int}},\star\})\, b \ . \tag{6.12}$$

Now from these infinitesimal transformations, we can get a finite transformation over ϕ by writing $\delta\phi = \phi/N$ and doing N infinitesimal transformations, while letting N tend to infinity. Mathematically this amounts to taking the limit

$$v \xrightarrow[\phi_0\to\phi_0+\phi]{} \lim_{N\to\infty}\left(1-\frac{\phi}{N}\{H_{\text{int}},\star\}\right)^N v \equiv e^{-\phi\{H_{\text{int}},\star\}} v = e^{\tau\{K_2,\star\}} v = e^{\tau P_2}\ v \ , \tag{6.13}$$

$$b \xrightarrow[\phi_0\to\phi_0+\phi]{} \lim_{N\to\infty}\left(1-\frac{\phi}{N}\{H_{\text{int}},\star\}\right)^N b \equiv e^{-\phi\{H_{\text{int}},\star\}} b = e^{\tau\{K_2,\star\}} b = e^{\tau P_2}\ b \ . \tag{6.14}$$

To actually compute this transformation we would like to use the spinors $|z\rangle = (z_1, \bar{z}_2)^T$ introduced in section 5.2.1, because we already know how they transform. Therefore we examine if we can write our generators in terms of those spinors. To see if this is possible, we simply equate our generators to those in (5.20),

$$K_+ = ve^{ib} \stackrel{!}{=} \bar{z_1}\bar{z_2} \ , \tag{6.15}$$

$$K_- = ve^{-ib} \stackrel{!}{=} z_1 z_2 \ , \tag{6.16}$$

$$K_3 = v \stackrel{!}{=} \frac{1}{2}(z_1\bar{z_1} + z_2\bar{z_2}) \ , \tag{6.17}$$

and we obtain the following general solution for the spinor variabels as a function of the phase space variables b and v:

$$z_1(b,v) = \sqrt{v}e^{-ib/2}e^{i\theta(b,v)} \ , \quad z_2(b,v) = \sqrt{v}e^{-ib/2}e^{-i\theta(b,v)} \ , \tag{6.18}$$

where $\theta(b,v)$ can be any real function of b and v. Our generators can thus be written as

$$K_+ = \bar{z_1}(b,v)\,\bar{z_2}(b,v) \ , \quad K_- = z_1(b,v)\,z_2(b,v) \ , \quad K_3 = \frac{1}{2}\left[\,z_1(b,v)\,\bar{z_1}(b,v) + z_2(b,v)\,\bar{z_2}(b,v)\,\right] \ . \tag{6.19}$$

We can thus extract the evolution of our generators from the evolution of the spinors. Then we can use the definition of the generators in terms of b and v to find the evolution of those variables. Using (5.30) and (5.15) and realizing that $\vec{u} = (0,\tau,0) \Rightarrow |\vec{u}|^2 = -\tau^2 < 0$, we have

$$|z(\tau)\rangle \equiv e^{\tau P_y}|z(0)\rangle = U_\tau|z(0)\rangle \ , \quad U_\tau = e^{i\tau\sigma_y} = \begin{pmatrix}\cosh(\tau/2) & \sinh(\tau/2)\\ \sinh(\tau/2) & \cosh(\tau/2)\end{pmatrix} \in \text{SU(1,1)} \ . \tag{6.20}$$

This provides us with two independent equations that we can solve for z_1 and for z_2. These can then be substituted in (6.4) using (6.19) to obtain the evolution of the generators

$$K_3(\tau) = K_3(0)\cosh(\tau) + K_1(0)\sinh(\tau) \ , \quad K_1(\tau) = K_3(0)\sinh(\tau) + K_1(0)\cosh(\tau) \ . \tag{6.21}$$

K_2 is, of course, constant, as it is the Hamiltonian. By choosing a different origin point in internal time, namely τ_0 such that $K_1(\tau_0) = 0$ (which is always possible), we write

$$K_3(\tau) = K_3(\tau_0)\cosh(\tau - \tau_0) \ , \quad K_1(\tau) = K_3(\tau_0)\sinh(\tau - \tau_0) \ , \tag{6.22}$$

and since $K_3 = v$ and $K_1 = v\cos(b)$, we deduce

$$v(\tau) = v_0 \cosh(\tau - \tau_0), \quad \cos(b) = \tanh(\tau - \tau_0). \tag{6.23}$$

These trajectories are identical to the ones (4.40) obtained by solving Hamilton's equations in the conventional way. Although the group theoretical approach might be less straightforward, it offers insight in structure of the system, the $\mathfrak{su}(1,1)$ structure, which may be useful in various ways. It offers e.g. a relatively simple way of quantizing the system; one can quantize the system by using the irreducible unitary representations of the transformation group SU(1,1) and this has been done in [7]. This, however, is beyond the purpose of this thesis. But with the $\mathfrak{su}(1,1)$ structure a new possibility opens up at the classical level as well. We can try to deform this structure and see if this leads to interesting new dynamics, like the dynamics generated by a nonzero cosmological constant.

7 Deformed FRW-Model

7.1 $\mathfrak{su}_q(1,1)$ Algebra of Phase Space Functions

In this section we use the the deformation procedure developed in section 5.3.2 to deform the Poisson algebra of phase space functions.
Step 1 is just the identification of the generators

$$K_+ = ve^{ib}, \quad K_- = ve^{-ib}, \quad K_3 = v, \tag{7.1}$$

as we already had in (6.2).
In **step 2**, we examine if we can write our generators in the form (5.20). But we have already seen that this is possible in section 6, namely for the spinor variables given by

$$z_1(b,v) = \sqrt{v}e^{-ib/2}e^{i\theta(b,v)}, \quad z_2(b,v) = \sqrt{v}e^{-ib/2}e^{-i\theta(b,v)} \ . \tag{6.18 revisited}$$

Our generators can thus be written as

$$K_+ = \bar{z_1}(b,v)\,\bar{z_2}(b,v), \quad K_- = z_1(b,v)\,z_2(b,v), \quad K_3 = \frac{1}{2}\left[\,z_1(b,v)\,\bar{z_1}(b,v) + z_2(b,v)\,\bar{z_2}(b,v)\,\right] \ . \tag{6.19 revisited}$$

Finally, **step 3** says that the generators can be deformed simply by replacing the spinor variables in $K_\pm$ by the deformed ones, (5.35) and (5.36), that are now given by

$$w_1 = (\gamma\sinh\gamma)^{-1/4}\sqrt{\frac{\sinh(\gamma v)}{v}}z_1 e^{i\gamma\alpha_1(v)} \tag{7.2}$$

$$w_2 = (\gamma\sinh\gamma)^{-1/4}\sqrt{\frac{\sinh(\gamma v)}{v}}z_2 e^{i\gamma\alpha_2(v)} \ . \tag{7.3}$$

Therefore the deformation is constituted by the replacements

$$K_+ = \bar{z_1}\bar{z_2} \quad \longrightarrow \quad Q_+ = \bar{w_1}\bar{w_2} \tag{7.4}$$

$$K_- = z_1 z_2 \quad \longrightarrow \quad Q_- = w_1 w_2 \tag{7.5}$$

$$K_3 = \frac{1}{2}(z_1\bar{z_1} + z_2\bar{z_2}) \longrightarrow \quad Q_3 = \frac{1}{2}(z_1\bar{z_1} + z_2\bar{z_2}) \ . \tag{7.6}$$

Substituting (7.2) and (7.3) (where we have imposed that $\alpha_1(v) = -\alpha_2(v) = 0$ for simplicity) we compute

$$Q_+ = (\gamma\sinh\gamma)^{-1/2}\sinh(\gamma v)e^{ib}, \quad Q_- = (\gamma\sinh\gamma)^{-1/2}\sinh(\gamma v)e^{-ib}, \quad Q_3 = v \ . \tag{7.7}$$

The deformed generators can also be written in terms of the quantum number $[v]_q$ of v as defined in (5.32) as

$$Q_+ = \sqrt{\frac{\sinh\gamma}{\gamma}}[v]_q e^{ib}, \quad Q_- = \sqrt{\frac{\sinh\gamma}{\gamma}}[v]_q e^{-ib}, \quad Q_3 = v \ . \tag{7.8}$$

Indeed, the Q_i close an $\mathfrak{su}_q(1,1)$ Poisson algebra:

$$\{Q_3, Q_\pm\} = \mp iQ_\pm, \quad \{Q_+, Q_-\} = i[2Q_3]_q \ . \tag{7.9}$$

In the limit $q \to 1$ ($\gamma \to 0$) (or by explicitly setting $q = 1$), we recover the original $\mathfrak{su}(1,1)$ generators[20].

We must admit that we have been a little lucky here. The deformation procedure that we have used assumes that the spinor variables have canonical Poisson brackets:

$$\{z_i, \bar{z_j}\} = -i\delta_{ij}, \quad \{z_1, z_2\} = 0 \ . \tag{7.10}$$

Our spinor variables (6.18) do not generally satisfy this assumption. It would therefore be natural to suggest that a particular choice of the function $\theta(b, v)$ might provide us with the correct, canonical Poisson brackets. And that, since the deformed generators do not depend on this function at all, this would ensure that the deformation procedure works correctly for any choice of the function. However, it is not possible to find even a single function $\theta(b, v)$ such that the Poisson brackets are canonical. Therefore it might have turned out that the deformed generators computed above would not have closed the deformed algebra.
Luckily they do though. And this is caused by the fact that actually the requirement that the spinor variables be canonical is a bit to strong. In principle it should be possible to derive a weaker set of requirements that the spinor variables should satisfy in order to ensure that both the original generators and the deformed generators as defined in respectively section 5.2.1 and 5.3.2 close the correct algebra. And it should be possible to find such a set of requirements, such that our spinor variables would satisfy those requirements.
Nevertheless, the requirement of the canonical Poisson brackets used in the procedure of section 5.3.2 is perfectly valid; if this requirement is satisfied, then the procedure will always work. The only thing is that it could be weakened. For our present purposes, however, this is not very important, since we have already found our deformed generators.

7.2 Deformed Hamiltonian and New Dynamics

Now that we have obtained the deformed generators, the rest is in principle straightforward. In the undeformed case the Hamiltonian is equal to the K_2 generator times a constant factor. In the deformed case we take the Hamiltonian to be the corresponding deformed generator, Q_2, times the same factor. It is given by

$$\frac{\widetilde{H}}{\sqrt{12\pi G}} = Q_2 \equiv \frac{1}{2i}(Q_+ - Q_-) = \sqrt{\frac{\sinh\gamma}{\gamma}}[v]_q \sin(b) = \frac{\sinh(\gamma v)\sin(b)}{\sqrt{\gamma \sinh\gamma}} \ . \tag{7.11}$$

Note that in the limit $q \to 1$ ($\gamma \to 0$) (or by explicitly setting $q = 1$), we recover the original Hamiltonian.
The evolution of b and v generated by the deformed Hamiltonian is then given by Hamilton's equations

$$\frac{\partial v}{\partial \tau} = \{v, Q_2\} = -\frac{\partial Q_2}{b} \tag{7.12}$$
$$\frac{\partial b}{\partial \tau} = \{b, Q_2\} = \frac{\partial Q_2}{v}. \tag{7.13}$$

[20]Since $[v]_q \to v$ and $\sinh(\gamma)/\gamma \to 1$ if $\gamma \to 0$.

Taking the derivatives, we get

$$\dot{v} \equiv \frac{\partial v}{\partial \tau} = -\frac{\sinh(\gamma v)\cos(b)}{\sqrt{\gamma \sinh(\gamma)}} \quad \dot{b} \equiv \frac{\partial b}{\partial \tau} = \frac{\gamma \cosh(\gamma v)\sin(b)}{\sqrt{\gamma \sinh(\gamma)}} \ . \tag{7.14}$$

Now we differentiate again to obtain two separated differential equations

$$\frac{\partial^2 v}{\partial \tau^2} \equiv \frac{\partial \dot{v}}{\partial \tau} = \frac{\partial \dot{v}}{\partial v}\frac{\partial v}{\partial \tau} + \frac{\partial \dot{v}}{\partial b}\frac{\partial b}{\partial \tau} = \frac{1}{2}\frac{\sinh(2\gamma v)}{\sinh(\gamma)} \tag{7.15}$$

$$\frac{\partial^2 b}{\partial \tau^2} \equiv \frac{\partial \dot{b}}{\partial \tau} = \frac{\partial \dot{b}}{\partial v}\frac{\partial v}{\partial \tau} + \frac{\partial \dot{b}}{\partial b}\frac{\partial b}{\partial \tau} = \frac{\gamma}{2}\frac{\sin(2b)}{\sinh(\gamma)} \ . \tag{7.16}$$

We have solved these equations using *Wolfram Mathematica*.

7.2.1 The Solution $v(\tau)$

The solution to the v equation, (7.15), is given by

$$v(\tau) = \pm\frac{i}{\gamma}\ \mathrm{am}\left(\frac{i\sqrt{\gamma}\sqrt{\mathrm{csch}(\gamma) + 4\gamma c_1}\tau}{\sqrt{2}}\middle| \frac{2}{4\gamma \sinh(\gamma) c_1 + 1}\right) \ . \tag{7.17}$$

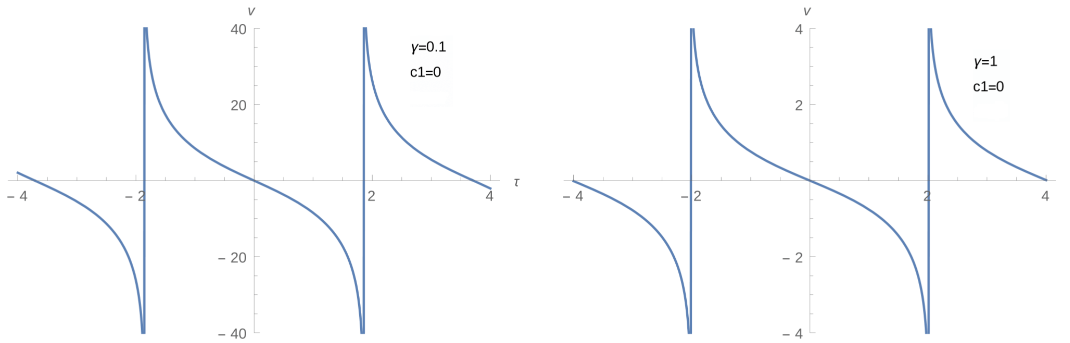
Here $\mathrm{am}(u|m)$ is the Jacobi amplitude function, $\mathrm{csch}(\gamma) = 1/\sinh(\gamma)$ and $c_1 \geq 0$ is an integration constant. We consider the (+) solution and we first examine the in which $c_1 = 0$. The solution is plotted for $\gamma = 0.1$ and $\gamma = 1$ in fig. 2a and fig. 2b, respectively. In these plots it is not clear that the maxima and minima of v are finite. This is the case though, and it is shown in figure 3. (The seeming irregularity of the maxima and minima is most likely due to numerical errors in the computation of the Jacobi amplitude function.) By plotting the solution for different values of γ we find that if $\gamma \leq 1$, then scaling γ corresponds almost exactly to (inversely) scaling the y-axis. More precisely: $\gamma \to \alpha\gamma$ corresponds to $v(\tau) \to v(\tau)/\alpha$, meaning that in these cases the Jacobi amplitude function is (nearly) independent of γ. Moreover, the solution is periodic for all $\gamma \leq 1$. For values of $\gamma \geq 1$ a rescaling of γ corresponds to an inverse rescaling of the y-axis and a scaling of the x-axis. Those scaling factors vary with the different values of γ. But the point is: for all these values of γ the overall shape of the curve is the same. For values of γ larger than 700, no solution could be found.
Now, if we keep γ constant, and we set c_1 to a positive nonzero value, this results in the x- and y-axis being rescaled and the curve being changed only slightly; the characteristic shape of the curve remains the same (fig. 4a and fig. 4b). We also note that rescaling $\gamma \to \alpha\gamma$ and $c_1 \to \alpha^{-2}c_1$ leads to a rescaling $v(\tau) \to v(\tau)/\alpha$.

7.2.2 The Solution $b(\tau)$

The solution to the b equation, (7.16), is given by

$$b(\tau) = \pm\mathrm{am}\left(\sqrt{(c_3 + \tau)^2(c_2 - \gamma\mathrm{csch}(\gamma))}| \frac{\gamma}{\gamma - c_2\sinh(\gamma)}\right) \ , \tag{7.18}$$

where c_2 and c_3 are integration constants. We can set $c_3 = 0$ to analyze the solutions, since a non-zero value will amount only to a translation in τ. Furthermore we redefine $c \to (\gamma\sinh(\gamma)c_2 + 1$



(a) (b)

Figure 2: Two plots of trajectories of $v(\tau)$ for $c_1 = 0$ and for different values of γ. The trajectories are generated by the deformed Hamiltonian. Note that the rescaling of γ amounts to an inverse rescaling in the y-axis with the same factor, meaning that $v(\tau) \propto 1/\gamma$.

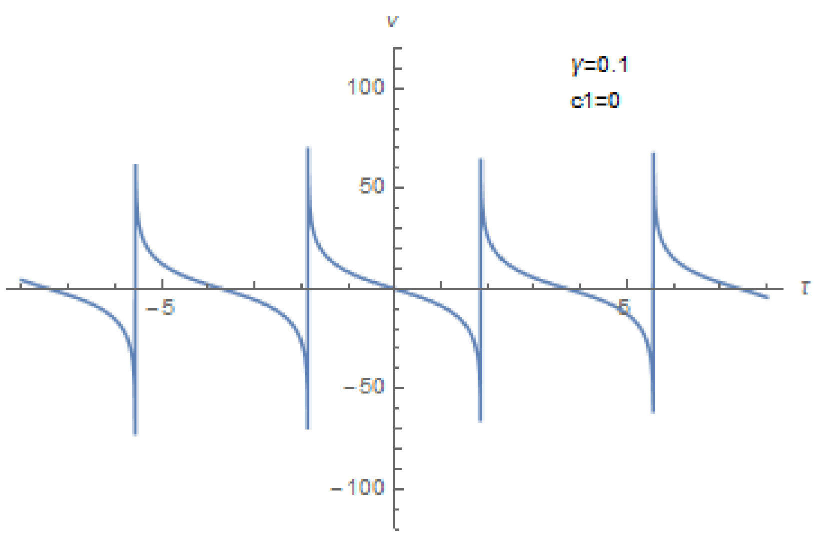
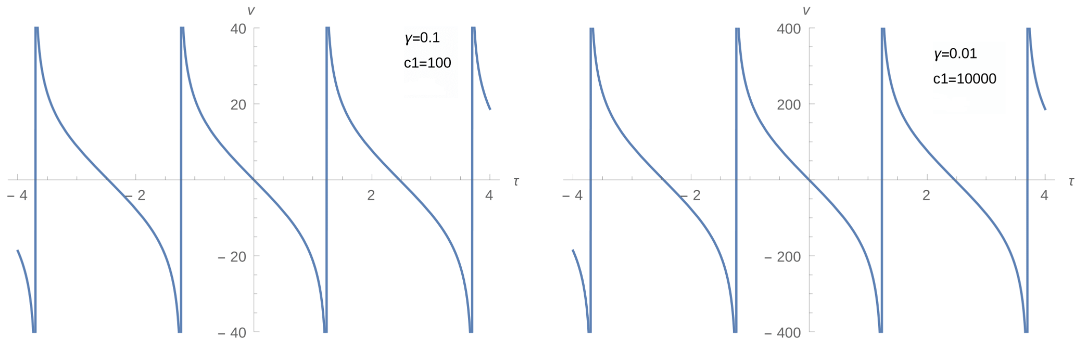


Figure 3: Plot of the trajectory of $v(\tau)$ for $\gamma = 0.1$ and $c_1 = 0$. The trajectory is generated by the deformed Hamiltonian. From this plot it is clear that the maxima and minima of v are finite.



(a)

(b)

Figure 4: Two plots of trajectories of $v(\tau)$ for $\gamma = 0.1$ and different values of c_1. The trajectories are generated by the deformed Hamiltonian. It evident that, in this this case for $\alpha = 0.1$, a rescaling $\gamma \to \alpha\gamma$ and $c_1 \to \alpha^{-2} c_1$ leads to a rescaling $v(\tau) \to v(\tau)/\alpha$.

to obtain the solution in a simpler form

$$b(\tau) = \pm \mathrm{am}\left(\sqrt{\gamma c \tau^2 \operatorname{csch}(\gamma)} \middle| \frac{-1}{c}\right). \tag{7.19}$$

For physically relevant solutions we must have $c > 0$, for otherwise the solution has a non-vanishing imaginary part. The overall characteristics of all the ($c > 0$) solutions are the same. Figure 5 shows three cases.

7.2.3 Cosmological Evolution in the Deformed Model

Now that we have the trajectories of b and v, the quality of interest is the volume of the cell $\mathcal{V}$ that, due to homogeneity, represents the volume of the universe. It is related to v by $V \propto |v|$. The trajectory of the volume is plotted in figure 6. The main characteristics of the trajectory of the volume are independent for the different values of the parameters, so this plot is representative for all cases. If we start, say, at $\tau = 0$, and we look at the cosmological evolution, we see the following. The universe starts with a Big Bang singularity, it then expands until it reaches a maximal volume, and then abruptly[21] it starts contracting again to eventually collapse in a Big Crunch singularity. Then the process repeats. (The seeming irregularity of the peaks we assume to be due to numerical errors.)

7.2.4 Comparison and Discussion

We have succeeded in finding a deformed (internal) Hamiltonian (7.11) that corresponds with a deformation of the $\mathfrak{su}(1,1)$ Poisson algebra on the phase space of effective sLQC. In the best scenario we would have been able to identify this Hamiltonian as the (internal) Hamiltonian (see appendix C of effective LQC with a nonzero cosmological constant. The two Hamiltonians are, however, clearly

[21] We do not have a well defined relation between proper time t and τ in the deformed model, so it is possible that this change is only abrupt in τ, and not in t.

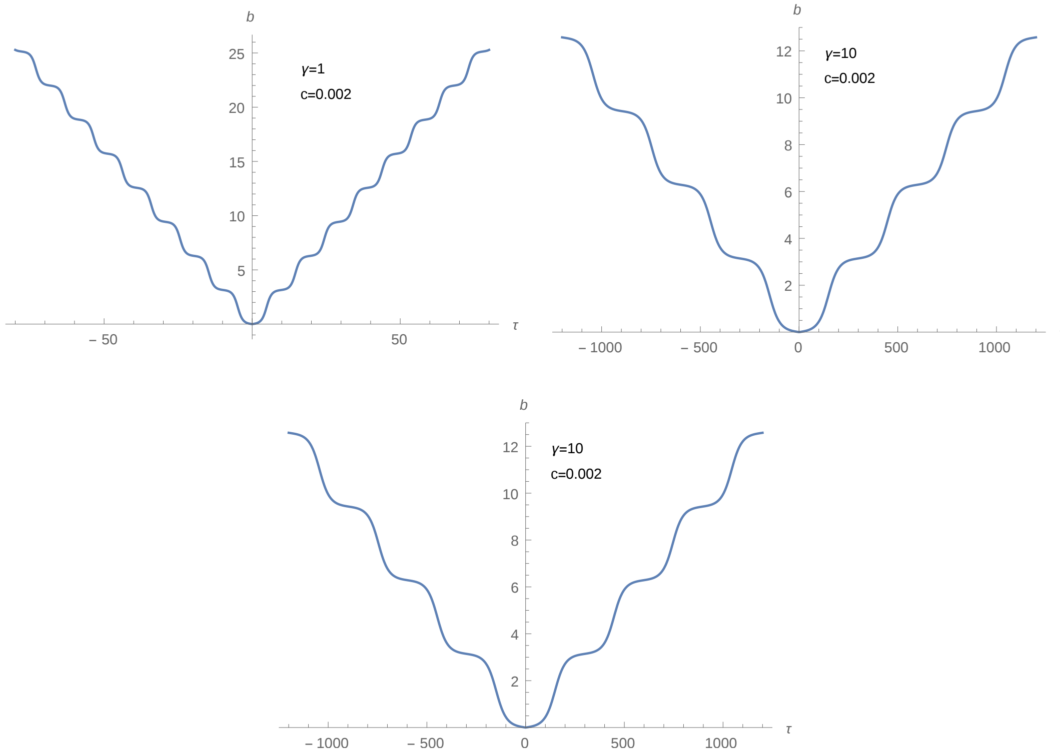


Figure 5: Three plots of trajectories of $b(\tau)$ generated by the deformed Hamiltonian for different values of γ and c. The curves are all different, but the overall characteristics are the same.

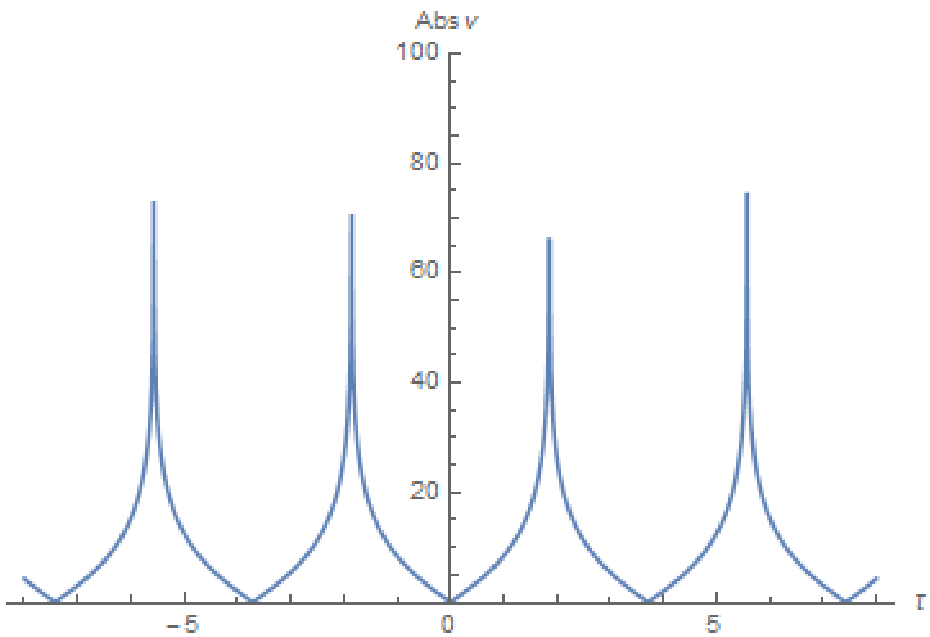
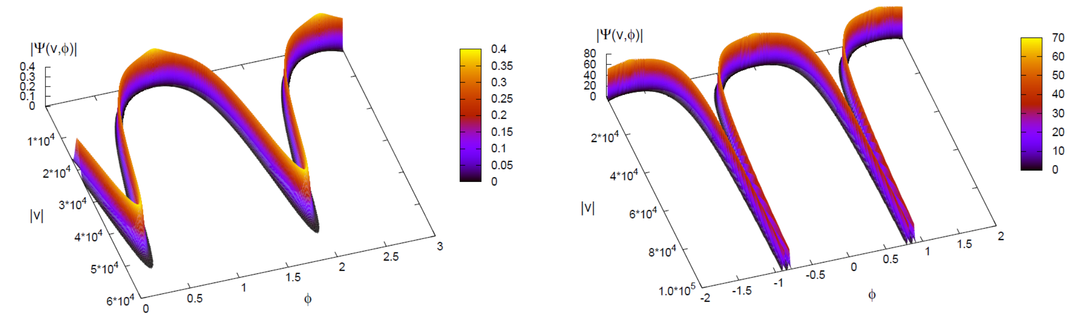


Figure 6: Trajectory of the absolute value of v, which is proportional to the volume of the cell $\mathcal{V}$, representing the volume of the universe. The trajectory is generated by the deformed Hamiltonian. The plot shows that the universe starts with a Big Bang singularity (e.g. $\tau = 0$), expands until it reaches a maximal volume, and then contracts again to eventually collapse in a Big Crunch singularity. Then the process repeats.



(a) (b)

Figure 7: The trajectories of the expectation values of the volume operator in the flat FRW model coupled to a scalar field in LQC, for a (a) negative cosmological constant (b) positive cosmological constant. The first ones to produce plots like these were Ashtekar A., Pawlowski T. and Singh P. The ones that are shown here are taken from [17].

different. We not recognize the deformed Hamiltonian as belonging to a different model within LQC either. What one might want to do in such a case is do a Taylor expansion in the deformation parameter γ to see if the Hamiltonians match for small values of γ. In this case, however, if we expand (7.11) in γ we expand automatically also in v, due to the term with $\sinh(\gamma v)$. Discarding high order terms in γ then means also discarding high order terms in v and hence the result will only be valid for small values of v (around the bounce).[22]

As opposed to comparing the Hamiltonians, we can as well compare the trajectory of the volume in the deformed model (fig. 6) with its trajectory in LQC with non-vanishing cosmological constant (fig. 7). Our trajectories do not match these trajectories exactly, but there are definitely similarities. The trajectories are all (nearly) periodic and bound from above and from below, and they share the same characteristics; their overall shape is very similar. However, in our model the minimum volume is zero, whereas in LQC it is non-zero (the singularity is resolved). Also, in the LQC model, the trajectories are smooth at the points of maximum and minimum volume, whereas in our case we have a discontinuity at minimum volume and an abrupt change at maximum volume as well, that looks more abrupt than in LQC.

We must conclude that we cannot encode the cosmological constant in the effective sLQC model through the $\mathfrak{su}$(1,1) deformation of the Poisson algebra of phase space functions, developed in this thesis. The realization of the deformation that we used is, however, not unique. In appendix D we give an example of an alternative realization of $\mathfrak{su}_q(1,1)$ on the effective sLQC phase space, resulting in a different deformed Hamiltonian. Although that Hamiltonian does not generate the desired dynamics either, it illustrates and sresses the fact that different realizations exist. This fact, together with the presence of the discussed similarities between the (main) deformed model and genuine LQC, should be seen, in my opinion, as an encouragement to further investigate the realizations of the deformation of the $\mathfrak{su}$(1,1) Poisson algebra on the effective sLQC phase space, to try to find one through which we can in fact encode the cosmological constant in the model.

[22] And even if we do this expansion anyway, the terms do not match.

APPENDIX

A The Covariant Derivative

The action of the covariant derivative ∇_ρ on a general tensor $T^{\mu_1,\dots,\mu_m}{}_{\nu_1,\dots,\nu_n}$ is defined as

$$\begin{aligned}
\nabla_\rho T^{\mu_1,\dots,\mu_m}{}_{\nu_1,\dots,\nu_n} \equiv T^{\mu_1,\dots,\mu_m}{}_{\nu_1,\dots,\nu_n;\rho} \equiv \partial_\rho T^{\mu_1,\dots,\mu_m}{}_{\nu_1,\dots,\nu_n} \\
+ \sum_{k=1}^{m} \Gamma^{\mu_k}{}_{\rho\sigma} T^{\mu_1,\dots,\mu_{k-1},\sigma,\mu_{k+1},\dots,\mu_m}{}_{\nu_1,\dots,\nu_n} \\
- \sum_{k=1}^{n} \Gamma^{\sigma}{}_{\rho\nu_k} T^{\mu_1,\dots,\mu_m}{}_{\nu_1,\dots,\nu_{k-1},\sigma,\nu_{k+1},\dots,\nu_n} \ . \qquad \text{(A.1)}
\end{aligned}$$

This translates e.g. into the more practical statements

$$\nabla_\rho T^\mu \equiv T^\mu{}_{;\rho} \equiv \partial_\rho T^\mu + \Gamma^\mu{}_{\rho\sigma} T^\sigma \ , \qquad \text{(A.2)}$$

$$\nabla_\rho T_\nu \equiv T_{\nu;\rho} \equiv \partial_\rho T_\nu - \Gamma^\sigma{}_{\rho\nu} T_\sigma \ , \qquad \text{(A.3)}$$

$$\nabla_\rho T^\mu{}_\nu \equiv T^\mu{}_{\nu;\rho} \equiv \partial_\rho T^\mu{}_\nu + \Gamma^\mu{}_{\rho\sigma} T^\sigma{}_\nu - \Gamma^\sigma{}_{\rho\nu} T^\mu{}_\sigma \ , \qquad \text{(A.4)}$$

where $\Gamma^\rho{}_{\mu\nu}$ is the Chirstoffel symbol (see section 2.2).
The spatial covariant derivative D_c associated with an induced spatial metric h_{ab} is defined as

$$D_c T^{a_1\cdots a_k}{}_{b_1\cdots b_l} = h^{a_1}{}_{d_1} \cdots h_{b_l}{}^{e_l} h_c{}^f \nabla_f T^{d_1\cdots d_k}{}_{e_1\cdots e_l} \ . \qquad \text{(A.5)}$$

This spatial covariant derivative is uniquely determined by h_{ab} such that is satisfies $D_a h_{bc} = 0$.

B The Poisson Bracket

The Poisson bracket of two phase space functions f and g is defined as

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial x_i} \frac{\partial f}{\partial p_i} \right) \ , \qquad \text{(B.1)}$$

where the sum runs over all pairs of (generalized) coordinates and their conjugate momenta. The Poisson bracket has the following properties for phase space functions f, g, and h:

$$\{f, g\} = -\{g, f\} \quad \text{(antisymmetry)}, \qquad \text{(B.2)}$$

$$\{f + g, h\} = \{f, h\} + \{g, h\} \quad \text{(linearity)}, \qquad \text{(B.3)}$$

$$\{fg, h\} = \{f, h\}g + f\{g, h\} \quad \text{(Leibniz rule)}, \qquad \text{(B.4)}$$

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \quad \text{(Jacobi identity)}. \qquad \text{(B.5)}$$

Furthermore there is the following property.

Proposition 1. For any complete set of phase space functions $u_k(\{x_i, p_i\}), v_k(\{x_i, p_i\})$ we have

$$\{f, g\} = \sum_{k,l} \left(\frac{\partial f}{\partial u_k} \frac{\partial g}{\partial v_l} - \frac{\partial g}{\partial u_k} \frac{\partial f}{\partial v_l} \right) \{u_k, v_l\} \ . \qquad \text{(B.6)}$$

Proof. We can write

$$df = \sum_k \left(\frac{\partial f}{\partial u_k} du_k + \frac{\partial f}{\partial v_k} du_k \right) \quad \Rightarrow \quad \frac{\partial f}{\partial x_i} = \sum_k \left(\frac{\partial f}{\partial u_k} \frac{\partial u_k}{\partial x_i} + \frac{\partial f}{\partial v_k} \frac{\partial u_k}{\partial x_i} \right) , \tag{B.7}$$

and analogously for $\partial f/\partial p_i$, $\partial g/\partial x_i$ and $\partial g/\partial p_i$. Therefore we can write

$$\begin{aligned} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial p_i} &= \sum_{k,l} \left(\frac{\partial f}{\partial u_k} \frac{\partial u_k}{\partial x_i} + \frac{\partial f}{\partial v_k} \frac{\partial u_k}{\partial x_i} \right) \left(\frac{\partial g}{\partial u_l} \frac{\partial u_l}{\partial p_i} + \frac{\partial g}{\partial v_l} \frac{\partial u_l}{\partial p_i} \right) \\ = \sum_{k,l} \left(\frac{\partial f}{\partial u_k}\frac{\partial u_k}{\partial x_i}\frac{\partial g}{\partial u_l}\frac{\partial u_l}{\partial p_i} + \frac{\partial f}{\partial u_k}\frac{\partial u_k}{\partial x_i}\frac{\partial g}{\partial v_l}\frac{\partial v_l}{\partial p_i} \right. & \left. + \frac{\partial f}{\partial v_k}\frac{\partial v_k}{\partial x_i}\frac{\partial g}{\partial u_l}\frac{\partial u_l}{\partial p_i} + \frac{\partial f}{\partial v_k}\frac{\partial v_k}{\partial x_i}\frac{\partial g}{\partial v_l}\frac{\partial v_l}{\partial p_i} \right) , \end{aligned} \tag{B.8}$$

$$\begin{aligned} \text{and} \quad \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial p_i} &= \sum_{k,l} \left(\frac{\partial g}{\partial u_k} \frac{\partial u_k}{\partial x_i} + \frac{\partial g}{\partial v_k} \frac{\partial u_k}{\partial x_i} \right) \left(\frac{\partial f}{\partial u_l} \frac{\partial u_l}{\partial p_i} + \frac{\partial f}{\partial v_l} \frac{\partial u_l}{\partial p_i} \right) \\ = \sum_{k,l} \left(\frac{\partial g}{\partial u_k}\frac{\partial u_k}{\partial x_i}\frac{\partial f}{\partial u_l}\frac{\partial u_l}{\partial p_i} + \frac{\partial g}{\partial u_k}\frac{\partial u_k}{\partial x_i}\frac{\partial f}{\partial v_l}\frac{\partial v_l}{\partial p_i} \right. & \left. + \frac{\partial g}{\partial v_k}\frac{\partial v_k}{\partial x_i}\frac{\partial f}{\partial u_l}\frac{\partial u_l}{\partial p_i} + \frac{\partial g}{\partial v_k}\frac{\partial v_k}{\partial x_i}\frac{\partial f}{\partial v_l}\frac{\partial v_l}{\partial p_i} \right) . \end{aligned} \tag{B.9}$$

The Poisson bracket of f and g is then obtained by subtracting (B.9) from (B.8) and summing over i. The resulting expression is

$$\begin{aligned} \{f,g\} = \sum_{i,k,l} \Bigg(& \frac{\partial f}{\partial u_k}\frac{\partial u_k}{\partial x_i}\frac{\partial g}{\partial u_l}\frac{\partial u_l}{\partial p_i} + \frac{\partial f}{\partial u_k}\frac{\partial u_k}{\partial x_i}\frac{\partial g}{\partial v_l}\frac{\partial v_l}{\partial p_i} + \frac{\partial f}{\partial v_k}\frac{\partial v_k}{\partial x_i}\frac{\partial g}{\partial u_l}\frac{\partial u_l}{\partial p_i} + \frac{\partial f}{\partial v_k}\frac{\partial v_k}{\partial x_i}\frac{\partial g}{\partial v_l}\frac{\partial v_l}{\partial p_i} \\ & - \frac{\partial g}{\partial u_k}\frac{\partial u_k}{\partial x_i}\frac{\partial f}{\partial u_l}\frac{\partial u_l}{\partial p_i} - \frac{\partial g}{\partial u_k}\frac{\partial u_k}{\partial x_i}\frac{\partial f}{\partial v_l}\frac{\partial v_l}{\partial p_i} - \frac{\partial g}{\partial v_k}\frac{\partial v_k}{\partial x_i}\frac{\partial f}{\partial u_l}\frac{\partial u_l}{\partial p_i} - \frac{\partial g}{\partial v_k}\frac{\partial v_k}{\partial x_i}\frac{\partial f}{\partial v_l}\frac{\partial v_l}{\partial p_i} \Bigg) . \end{aligned} \tag{B.10}$$

By relabeling dummy indices, we see that the 1st term cancels the 5th, and the 4th term cancels the 8th, so we can write

$$\begin{aligned} \{f,g\} &= \sum_{i,k,l} \left(\frac{\partial f}{\partial u_k}\frac{\partial u_k}{\partial x_i}\frac{\partial g}{\partial v_l}\frac{\partial v_l}{\partial p_i} + \frac{\partial f}{\partial v_k}\frac{\partial v_k}{\partial x_i}\frac{\partial g}{\partial u_l}\frac{\partial u_l}{\partial p_i} - \frac{\partial g}{\partial u_k}\frac{\partial u_k}{\partial x_i}\frac{\partial f}{\partial v_l}\frac{\partial v_l}{\partial p_i} - \frac{\partial g}{\partial v_k}\frac{\partial v_k}{\partial x_i}\frac{\partial f}{\partial u_l}\frac{\partial u_l}{\partial p_i} \right) \\ &= \sum_{i,k,l} \left[\left(\frac{\partial u_k}{\partial x_i}\frac{v_l}{\partial p_i} \right) \left(\frac{\partial f}{\partial u_k}\frac{\partial g}{\partial v_l} - \frac{\partial g}{\partial u_k}\frac{\partial f}{\partial v_l} \right) - \left(\frac{\partial v_k}{\partial x_i}\frac{u_l}{\partial p_i} \right) \left(\frac{\partial g}{\partial v_k}\frac{\partial f}{\partial u_l} - \frac{\partial f}{\partial v_k}\frac{\partial g}{\partial u_l} \right) \right] . \end{aligned} \tag{B.11}$$

Now we explicitly interchange the (dummy) labels k and l in the right half of the last line of this equation. Then we obtain

$$\begin{aligned} \{f,g\} &= \sum_{i,k,l} \left[\left(\frac{\partial u_k}{\partial x_i}\frac{v_l}{\partial p_i} \right) \left(\frac{\partial f}{\partial u_k}\frac{\partial g}{\partial v_l} - \frac{\partial g}{\partial u_k}\frac{\partial f}{\partial v_l} \right) - \left(\frac{\partial v_l}{\partial x_i}\frac{u_k}{\partial p_i} \right) \left(\frac{\partial g}{\partial v_l}\frac{\partial f}{\partial u_k} - \frac{\partial f}{\partial v_l}\frac{\partial g}{\partial u_k} \right) \right] \\ &= \sum_{i,k,l} \left[\left(\frac{\partial u_k}{\partial x_i}\frac{v_l}{\partial p_i} \right) \left(\frac{\partial f}{\partial u_k}\frac{\partial g}{\partial v_l} - \frac{\partial g}{\partial u_k}\frac{\partial f}{\partial v_l} \right) - \left(\frac{u_k}{\partial p_i}\frac{\partial v_l}{\partial x_i} \right) \left(\frac{\partial f}{\partial u_k}\frac{\partial g}{\partial v_l} - \frac{\partial g}{\partial u_k}\frac{\partial f}{\partial v_l} \right) \right] \\ &= \sum_{k,l} \left(\frac{\partial f}{\partial u_k}\frac{\partial g}{\partial v_l} - \frac{\partial g}{\partial u_k}\frac{\partial f}{\partial v_l} \right) \sum_i \left(\frac{\partial u_k}{\partial x_i}\frac{v_l}{\partial p_i} - \frac{u_k}{\partial p_i}\frac{\partial v_l}{\partial x_i} \right) \qquad \text{(B.12)} \\ &= \sum_{k,l} \left(\frac{\partial f}{\partial u_k}\frac{\partial g}{\partial v_l} - \frac{\partial g}{\partial u_k}\frac{\partial f}{\partial v_l} \right) \{u_k, v_l\} , \qquad \text{(B.13)} \end{aligned}$$

which is what we wanted to prove. □

Corollary. For a two-dimensional phase space we have

$$\{f, g\} = \left(\frac{\partial f}{\partial u}\frac{\partial g}{\partial v} - \frac{\partial g}{\partial u}\frac{\partial f}{\partial v}\right)\{u, v\} \tag{B.14}$$

for any complete pair of phase space functions.

B.1 From Poisson (Bracket) Algebras to (Commutator) Lie Algebras

If one has a Poisson algebra, then in many cases one can construct a corresponding Lie algebra, where the Lie bracket is given by the commutators. This is the result of the following proposition.

Proposition 2. Assume that we have phase space functions A and B that have the Poisson bracket $\{A, B\} = C$, for another phase space function C. Then the commutator of the operators $P_A = \{A, \star\}$ and $P_B = \{B, \star\}$ is given by $[P_A, P_B] = P_C$, where $P_C = \{C, \star\}$.

Proof. We let the operators act on a scalar variable x and begin by writing out

$$P_B x = \{B, \star\}x = \{B, x\} = \sum_i \left(\frac{\partial B}{\partial x_i}\frac{\partial x}{\partial p_i} - \frac{\partial B}{\partial p_i}\frac{\partial x}{\partial x_i}\right) , \tag{B.15}$$

$$P_A x = \{A, \star\}x = \{A, x\} = \sum_i \left(\frac{\partial A}{\partial x_i}\frac{\partial x}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial x}{\partial x_i}\right) . \tag{B.16}$$

Then we can write

$$\begin{aligned} P_A P_B x = \{A, P_B x\} &= \left\{A, \sum_l \left(\frac{\partial B}{\partial x_l}\frac{\partial x}{\partial p_l} - \frac{\partial B}{\partial p_l}\frac{\partial x}{\partial x_l}\right)\right\} \\ = \sum_{k,l} \Bigg(&\frac{\partial A}{\partial x_k}\frac{\partial^2 B}{\partial p_k x_l}\frac{\partial x}{\partial p_l} + \frac{\partial A}{\partial x_k}\frac{\partial B}{\partial x_l}\frac{\partial^2 x}{\partial p_l \partial p_k} - \frac{\partial A}{\partial x_k}\frac{\partial^2 B}{\partial p_k \partial p_l}\frac{\partial x}{\partial x_l} - \frac{\partial A}{\partial x_k}\frac{\partial B}{\partial p_l}\frac{\partial^2 x}{\partial x_l \partial p_k} \\ &- \frac{\partial A}{\partial p_k}\frac{\partial^2 B}{\partial x_k \partial x_l}\frac{\partial x}{\partial p_l} - \frac{\partial A}{\partial p_k}\frac{\partial B}{\partial x_l}\frac{\partial^2 x}{\partial x_k \partial p_l} + \frac{\partial A}{\partial p_k}\frac{\partial^2 B}{\partial x_k \partial p_l}\frac{\partial x}{\partial x_l} + \frac{\partial A}{\partial p_k}\frac{\partial B}{\partial p_l}\frac{\partial^2 x}{\partial x_l \partial x_k}\Bigg) . \end{aligned} \tag{B.17}$$

And for $P_B P_A x$ we get the same expression but with A and B interchanged. We can therefore write

$$[P_A, P_B]x = P_A P_B x - P_B P_A x$$

$$= \sum_{k,l} \left(\frac{\partial A}{\partial x_k} \frac{\partial^2 B}{\partial p_k x_l} \frac{\partial x}{\partial p_l} + \frac{\partial A}{\partial x_k} \frac{\partial B}{\partial x_l} \frac{\partial^2 x}{\partial p_l \partial p_k} - \frac{\partial A}{\partial x_k} \frac{\partial^2 B}{\partial p_k \partial p_l} \frac{\partial x}{\partial x_l} - \frac{\partial A}{\partial x_k} \frac{\partial B}{\partial p_l} \frac{\partial^2 x}{\partial x_l \partial p_k} \right.$$
$$- \frac{\partial A}{\partial p_k} \frac{\partial^2 B}{\partial x_k \partial x_l} \frac{\partial x}{\partial p_l} - \frac{\partial A}{\partial p_k} \frac{\partial B}{\partial x_l} \frac{\partial^2 x}{\partial x_k \partial p_l} + \frac{\partial A}{\partial p_k} \frac{\partial^2 B}{\partial x_k \partial p_l} \frac{\partial x}{\partial x_l} + \frac{\partial A}{\partial p_k} \frac{\partial B}{\partial p_l} \frac{\partial^2 x}{\partial x_l \partial x_k}$$
$$- \frac{\partial B}{\partial x_k} \frac{\partial^2 A}{\partial p_k x_l} \frac{\partial x}{\partial p_l} - \frac{\partial B}{\partial x_k} \frac{\partial A}{\partial x_l} \frac{\partial^2 x}{\partial p_l \partial p_k} + \frac{\partial B}{\partial x_k} \frac{\partial^2 A}{\partial p_k \partial p_l} \frac{\partial x}{\partial x_l} + \frac{\partial B}{\partial x_k} \frac{\partial A}{\partial p_l} \frac{\partial^2 x}{\partial x_l \partial p_k}$$
$$\left. + \frac{\partial B}{\partial p_k} \frac{\partial^2 A}{\partial x_k \partial x_l} \frac{\partial x}{\partial p_l} + \frac{\partial B}{\partial p_k} \frac{\partial A}{\partial x_l} \frac{\partial^2 x}{\partial x_k \partial p_l} - \frac{\partial B}{\partial p_k} \frac{\partial^2 A}{\partial x_k \partial p_l} \frac{\partial x}{\partial x_l} - \frac{\partial B}{\partial p_k} \frac{\partial A}{\partial p_l} \frac{\partial^2 x}{\partial x_l \partial x_k} \right) . \tag{B.18}$$

By relabeling dummy indices we see that all terms with second derivatives of x cancel each other, as the 2nd term cancels the 10th term, the 4th term cancels the 14th, the 6th term cancels the 12th, and the 8th term cancels the 16th. Hence we have

$$[P_A, P_B]x = P_A P_B x - P_B P_A x$$

$$= \sum_{k,l} \left(\frac{\partial A}{\partial x_k} \frac{\partial^2 B}{\partial p_k x_l} \frac{\partial x}{\partial p_l} - \frac{\partial A}{\partial x_k} \frac{\partial^2 B}{\partial p_k \partial p_l} \frac{\partial x}{\partial x_l} - \frac{\partial A}{\partial p_k} \frac{\partial^2 B}{\partial x_k \partial x_l} \frac{\partial x}{\partial p_l} + \frac{\partial A}{\partial p_k} \frac{\partial^2 B}{\partial x_k \partial p_l} \frac{\partial x}{\partial x_l} \right.$$
$$\left. - \frac{\partial B}{\partial x_k} \frac{\partial^2 A}{\partial p_k x_l} \frac{\partial x}{\partial p_l} + \frac{\partial B}{\partial x_k} \frac{\partial^2 A}{\partial p_k \partial p_l} \frac{\partial x}{\partial x_l} + \frac{\partial B}{\partial p_k} \frac{\partial^2 A}{\partial x_k \partial x_l} \frac{\partial x}{\partial p_l} - \frac{\partial B}{\partial p_k} \frac{\partial^2 A}{\partial x_k \partial p_l} \frac{\partial x}{\partial x_l} \right) . \tag{B.19}$$

This is the LHS of the formula that we want to prove. For the RHS we use the relation $C = \{A, B\}$ and write out

$$P_C x = \{C, x\} = \{\{A, B\}, x\} = \sum_k \left\{ \frac{\partial A}{\partial x_k} \frac{\partial B}{\partial p_k} - \frac{\partial A}{\partial p_k} \frac{\partial B}{\partial x_k}, x \right\}$$
$$= \sum_{k,l} \left(\frac{\partial^2 A}{\partial x_k \partial x_l} \frac{\partial B}{\partial p_k} \frac{\partial x}{\partial p_l} + \frac{\partial A}{\partial x_k} \frac{\partial^2 B}{\partial p_k x_l} \frac{\partial x}{\partial p_l} - \frac{\partial^2 A}{\partial p_k \partial x_l} \frac{\partial B}{\partial x_k} \frac{\partial x}{\partial p_l} - \frac{\partial A}{\partial p_k} \frac{\partial^2 B}{\partial x_k \partial x_l} \frac{\partial x}{\partial p_l} \right.$$
$$\left. - \frac{\partial^2 A}{\partial x_k \partial p_l} \frac{\partial B}{\partial p_k} \frac{\partial x}{\partial x_l} - \frac{\partial A}{\partial x_k} \frac{\partial^2 B}{\partial p_k \partial p_l} \frac{\partial x}{\partial x_l} + \frac{\partial^2 A}{\partial p_k \partial p_l} \frac{\partial B}{\partial x_k} \frac{\partial x}{\partial x_l} + \frac{\partial A}{\partial p_k} \frac{\partial^2 B}{\partial x_k \partial p_l} \frac{\partial x}{\partial x_l} \right) . \tag{B.20}$$

Comparing (B.19) and (B.20), we see that the expressions are identical, and so we have

$$\forall x : \quad [P_A, P_B]x = P_C x \quad \Rightarrow \quad [P_A, P_B] = P_C \ , \tag{B.21}$$

which is what we wanted to prove. □

Corollary. If J_+, J_-, J_3 are phase space functions that form an $\mathfrak{su}(1,1)$ Poisson algebra, i.e.

$$\{J_3, J_\pm\} = \mp i J_\pm, \quad \{J_+, J_-\} = 2iJ_3 \ , \tag{B.22}$$

then the operators

$$P_+ = \{J_+, \star\}, \quad P_- = \{J_-, \star\}, \quad P_3 = \{J_3, \star\} \tag{B.23}$$

form an $\mathfrak{su}(1,1)$ Lie algebra with their commutators, i.e.

$$[P_3, P_\pm] = \mp i P_\pm, \quad [P_+, P_-] = 2iP_3 \ . \tag{B.24}$$

C Effective Internal Hamiltonian for a Nonzero Cosmological Constant

The gravitational (integrated) Hamiltonian Constraint for the ($k = 0$, $\Lambda = 0$) FRW-model, coupled to a scalar field reads

$$C = \frac{p_\phi^2}{8\pi G|v|} - \frac{3}{2}b^2|v| = 0 \ , \tag{C.1}$$

When the cosmological constant Λ is nonzero (but still $k = 0$), this constraint gets an additional term and it is modified to

$$C = \frac{p_\phi^2}{8\pi G|v|} - \frac{3}{2}b^2|v| + \frac{1}{2}\Lambda|v| = 0 \ , \tag{C.2}$$

The effective dynamics is obtained by replacing $b \to \sin(\lambda b)/\lambda$ for a constant $\lambda \in \mathbb{R}$. The effective constraint then reads

$$C_{(\mathrm{eff})} = -\frac{3}{2}\frac{\sin^2(\lambda b)}{\lambda^2}|v| + \frac{p_\phi^2}{8\pi G|v|} + \frac{1}{2}\Lambda|v| = 0 \ , \tag{C.3}$$

Just as we did in the case of a vanishing cosmological constant, we change variables, according to the canonical transformation $v \to v' = v/\lambda$, $b \to b' = \lambda b$, and redefine $v := v'$, $b := b'$, the constraint then reads

$$C_{(\mathrm{eff})} = -\frac{3}{2}\frac{\sin^2(b)}{|\lambda|}|v| + \frac{p_\phi^2}{8\pi G|\lambda||v|} + \frac{1}{2}\Lambda|\lambda||v| = 0 \ . \tag{C.4}$$

By inverting this relation we find the expression of the internal Hamiltonian (the scalar field momentum),

$$H_{\mathrm{int}} = p_\phi = \pm\sqrt{12\pi G}v\sqrt{\sin^2(b) - \tfrac{1}{3}\lambda^2\Lambda} \ . \tag{C.5}$$

The evolution (in ϕ) that this Hamiltonian generates is an excellent approximation to the genuine quantum dynamics with a nonzero cosmological constant.

D Alternative Deformation

The realization of $\mathfrak{su}_q(1,1)$ that we have established on the effective LQC phase phase is not unique. In this section we establish a different realization of the algebra and show that this leads to a different deformed Hamiltonian. Although this Hamiltonian will prove to be a complex function, it illustrates the fact that the deformed dynamics that we have found in this thesis is not the only option. It emphasizes the possibility that there might be an even different realization out there, that would in fact lead to the inclusion of a cosmological constant in the model under consideration.

D.1 Alternative Realization of $\mathfrak{su}_q(1,1)$

$\mathfrak{su}$(1,1) can be realized in terms two complex variables a_+ and a_- that satisfy the Poisson bracket

$$\{a_+, a_-\} = -i. \tag{D.1}$$

Writing $N = a_+a_-$, the $\mathfrak{su}$(1,1) Poisson algebra is realized by the generators

$$K_+ = a_+a_+a_- = Na_+ \ , \quad K_- = a_- \ , \quad K_3 = a_+a_- = N \ , \tag{D.2}$$

that satisfy (5.18). To deform the algebra we take the following ansatz

$$Q_+ = F(N)a_+ \ , \quad Q_- = F(N)a_- \ , \quad Q_3 = K_3 = N \ , \tag{D.3}$$

for any function F(N) of N. We then automatically have the correct Poisson brackets $\{Q_3, Q_\pm\} = \mp iQ_\pm$. To get the right function F(N) we impose the other Poisson bracket of the algebra:

$$i[2K_3]_q = \{Q_+, Q_-\} = a_+F(N)\{F(N), a_-\} + a_-F(N)\{a_+, F(N)\} + iF(N)^2 \ . \tag{D.4}$$

Using the property of the Poisson bracket (B.14) and multiplying both sides by $-i$ this is equivalent to

$$[2N]_q = 2NF\frac{\partial F}{\partial N} + F^2. \tag{D.5}$$

This equation can be simplified by writing $F = \sqrt{G}$, leading to

$$[2N]_q = N\frac{\partial G}{\partial N} + G \ , \tag{D.6}$$

and next we write $\widetilde{G} = (2\sinh(\gamma))^{-1}G$ to obtain the equation in its final form

$$q^{2N} - q^{-2N} = N\frac{\partial \widetilde{G}}{\partial N} + \widetilde{G} \ . \tag{D.7}$$

The equation is solved by

$$\widetilde{G}(N) = \frac{c_1}{N} + \frac{q^{2N} + q^{-2N}}{2Nln(q)} \ , \tag{D.8}$$

where c_1 is an arbitrary integration constant. Setting $c_1 = 0$ for simplicity and translating this back to F(N) we have the solution

$$F(N) = \sqrt{\frac{q^{2N} + q^{-2N}}{2N ln(q)(q - q^{-1})}} = \sqrt{\frac{1}{2N\gamma}\frac{\cosh(2\gamma N)}{\sinh(\gamma)}} \ . \tag{D.9}$$

One can check that indeed the generators

$$Q_+ = \sqrt{\frac{1}{2N\gamma}\frac{\cosh(2\gamma N)}{\sinh(\gamma)}} a_+ \ , \quad Q_- = \sqrt{\frac{1}{2N\gamma}\frac{\cosh(2\gamma N)}{\sinh(\gamma)}} a_- \ , \quad Q_3 = N \tag{D.10}$$

close a $\mathfrak{su}_q(1,1)$ algebra.

D.2 Application to the Effective sLQC Phase Space

We will now apply the procedure developed above to the effective sLQC phase space. We start with our phase space functions

$$K_+ = ve^{ib} \ , \quad K_- = ve^{-ib} \ , \quad K_3 = v \ , \tag{D.11}$$

that form a $\mathfrak{su}$(1,1) Poisson algebra. We are now going to write the generators in the form (D.2). By equating the corresponding generators we find

$$a_+ = e^{ib} \ , \quad a_- = ve^{-ib} \ . \tag{D.12}$$

We can now write our generators in terms of a_+ and a_- and deform the algebra according to section D.1.

$$K_+ = a_+ a_+ a_- = N a_+ \quad \longrightarrow \quad Q_+ = F(N) a_+ \tag{D.13}$$

$$K_- = a_- \quad \longrightarrow \quad Q_- = F(N) a_- \tag{D.14}$$

$$K_3 = a_+ a_- = N \quad \longrightarrow \quad Q_3 = K_3 = N \tag{D.15}$$

Substituting everything, the deformed generators read

$$Q_+ = \sqrt{\frac{1}{2v\gamma}\frac{\cosh(2\gamma v)}{\sinh(\gamma)}} e^{ib} \ , \quad Q_- = \sqrt{\frac{1}{2v\gamma}\frac{\cosh(2\gamma v)}{\sinh(\gamma)}} ve^{-ib} \ , \quad Q_3 = v \tag{D.16}$$

and one can check that they indeed form a $\mathfrak{su}_q(1,1)$ algebra. Now we would like to define the deformed Hamiltonian as $\widetilde{H} = Q_y$, but we have a problem: since Q_+ and Q_- are not each others complex conjugate, $Q_y \in \mathbb{C}$ is not real, and a Hamiltonian should, of course, always be real, since it corresponds with the total energy of the system[23].

$$Q_y \equiv \frac{1}{2i}(Q_+ - Q_-) = \frac{1}{2i}\sqrt{\frac{1}{2v\gamma}\frac{\cosh(2\gamma v)}{\sinh(\gamma)}} \left(e^{ib} - ve^{-ib}\right) \ . \tag{D.17}$$

The Hamiltonian defined by this alternative realization of $\mathfrak{su}$(1,1) is therefore not a physical Hamiltonian and it doesn't generate any physical dynamics.

[23]In this case it actually corresponds to the momentum of the scalar field, since we're dealing with the *internal* Hamiltonian.

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