Radboud University Nijmegen

Theoretical High Energy Physics

Master Thesis

Advancements on the Covariance of the κ-Poincaré Model and Relative Locality

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Contents

Introduction

The field of quantum gravity can perhaps be described as the search for the holy grail of modern physics. Although Einstein's theory of General Relativity provides an extremely successful description of gravity on a wide range of scales[1], the fact that it describes gravity as a *classical* force is fundamentally inconsistent with the fact that matter (and everything else) in the universe behaves quantum mechanically. Indeed the classical description of gravity is naturally expected to break down at Planck scale energies $E_{\rm Pl} = \sqrt{\hbar c^5/G} \sim 10^{19}$ GeV.
Since such high energy scales are at present, as well as in the near future, virtually inaccessable in laboratory experiments, the traditional approaches to quantum gravity are guided almost exclusively by mathematical consistency. However, it became clear about two decades ago [2] that astronomical observations may in fact be used to probe quantum gravity effects, the reason being that the large distances that astronomical photons travel can lead to a significant magnification of *a priori* extremely small quantum gravity effects. A much studied such effect is a possible energy dependence of photon velocities. Such an energy dependence might be inferred by detecting differences in the arrival times of photons of different wavelengths that are emitted in gamma ray bursts [2, 3, 4, 5, 6, 7, 8]. This has led to the rise of an alternative research program, which goes under the name Quantum Gravity Phenomenology (see [9] for a review). It prioritizes the connection to these kinds of experiments and aims to construct phenomenological descriptions of quantum gravity effects that are in fact testable or might be

[1] General Relativity has been tested succesfully on distance scales ranging from 10^{-6} m [1] to astrophysical scales, (effects often attributed to) dark matter and dark energy being a possible signal that the theory might not work on larger scales. But note that there is a truly enormous gap between 10^{-6} m and the Planck length $\ell_{\rm Pl} = \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-35}$ m.

testable in the near future. Among the most promising such effects are departures from Poincaré /Lorentz invariance and modified energy-momentum dispersion relations[2].

In the realm of quantum gravity phenomenology an important role is played by semi-classical models, which use a classical framework – often Hamiltonian mechanics – to describe the physics, yet nevertheless contain Planck-scale modifications to the classical equations. We might say that these models correspond to the limiting case $\hbar \to 0$. If we send $G \to 0$ at the same time, quantum gravity effects can manifest through the ratio $\hbar/G$, corresponding to the Planck energy $E_{\mathrm{Pl}} = \sqrt{\hbar c^5/G}$, which is still a free parameter in this limit. We then arrive in the realm of deformations of Special Relativity, the deformation parameter being the Planck energy. Just as the speed of light is an invariant in Special Relativity, the Planck energy should be a second relativistic invariant in these deformed theories, which means that Poincaré transformations, and hence the infinitesimal symmetry algebra, must be modified as well in order to achieve this.

A very interesting and much-studied candidate for such a deformed symmetry algebra is the so-called κ-Poincaré Hopf algebra [10, 11, 12, 13]. Mathematically this is the most studied Hopf algebra deformation of the Poincaré algebra, and from a physics point of view there are compelling arguments suggesting that this algebra might emerge in certain limits of quantum gravity [14, 15, 16, 17][3]. Although the arguments are mainly based on the 2+1 (and 3)[4] dimensional case (with the exception of [16]), where gravity can be described by a topological field theory that has no dynamical degrees of freedom, the κ-Poincaré Hopf algebra is well-defined in any number of dimensions, so it is also very much of interest to study its implications for physics in 3+1 dimensions.

Since its discovery in the early 90's [10, 11, 12] the κ-Poincaré Hopf algebra has been the subject of a lot of research, often in connection with physics. A major advancement was the formulation of this Hopf algebra in the so-called bicrossproduct basis [13], which is presently its most encountered form. As shown in [21] the κ-Poincaré algebra, being a sufficiently nice so-called h-adic Hopf alge-

[2]Modified dispersion relations can in particular be related to the energy dependence of photon velocities.

[3]Other authors argue that it is not the κ-Poincaré algebra that emerges but some other deformed algebra [18, 19, 20].

[4]That is, Euclidean

bra, has the power to completely specify a semi-classical model of interacting particles – including modified dispersion relation, modified momentum conservation law and modified Poincaré transformations – that reduces to Special Relativity when the deformation parameter κ (which is identified with the Planck energy), tends to infinity (or, equivalently, when all energies and momenta considered are much smaller than the Planck scale). This model, which we refer to simply as *the κ-Poincaré model*, will be the main object of study in this thesis. It is very much relevant to quantum gravity phenomenology, as it makes a number of predictions (e.g. an energy-dependent photon velocity) that might actually be tested by the astronomical observations alluded to above.

Before going into more detail about the model and its role in this thesis, let us stress one of its important features, namely that spacetime locality becomes relative to the observer. What this means is that when particles interact with each other then, depending on the observer, the particles' worldlines need not meet at a single point in spacetime. Locality of an interaction is only guaranteed for observers that are close[5] to the interaction. For observers far away from the interaction it may look like the interaction is nonlocal. Of course, even though locality becomes dependent on the observer in this way, there still has to be an objective, observer-independent, account of the actual physics in each scenario, and this account is provided in phase space. Each particle has an objective trajectory in the phase space manifold, although the *coordinates* of this trajectory (momentum coordinates as well as spacetime coordinates) may depend on the observer[6]. These ideas have been dubbed Relative Locality and are neatly described by the formalism proposed recently in [22], which generally goes under the same name. To distinguish the general idea of the relativity of locality and the specific formalism of [22] we will refer to the latter as *the Relative Locality framework* (RLF) throughout this thesis. The RLF connects physical properties such as the relativity of locality to the geometry of momentum space, allowing for momentum space to be a curved manifold instead of merely the linear tan-

[5]The term *close* needs to be defined clearly for this statement to make sense. We will defer this to chapter 2.

[6]This is very similar to the fact in Special Relativity that different observers will always agree on the (abstract) spacetime *point* in the spacetime manifold where an interaction happens, even though they will usually not agree on the spacetime *coordinates*, because each observer has its own coordinate system.

gent space to the spacetime manifold [23]. Indeed, spacetime is now described as the cotangent space to momentum space, and phase space is by definition the corresponding cotangent bundle. Where General Relativity can be thought of as the $\hbar \to 0, M_{Pl} \to 0$ limit of a fundamental theory of quantum gravity, while G remains nonvanishing, the RLF can be thought of as (a candidate for) the $\hbar \to 0, G \to 0$ limit of quantum gravity, while instead the Planck mass remains finite. General Relativity deals with curved spacetime and flat momentum space; the RLF deals with curved momentum space and flat spacetime. This realizes a kind of duality between spacetime and momentum space that was already envisioned by Max Born in 1938 [24] as a prerequisite for being able to unite General Relativity with Quantum Mechanics.
Now let us return to the earlier mentioned paper [21]. Here the κ-Poincaré model was derived and it was shown that it corresponds to a de Sitter momentum space geometry in the context of the RLF. One of the most counter-intuitive results of the paper was that the deformed boosts [25], induced by the κ-Poincaré algebra, must act on interacting particles in a nontrivial way: particles do not necessarily transform with the same boost rapidity. Instead, the boost rapidity of each particle that participates in a given interaction vertex is influenced (via a so-called *backreaction*) by the momenta of other particles in that vertex. The explicit form of this backreaction is determined by the requirement that the deformed momentum conservation law behaves covariantly under the deformed boosts (or equivalently, that the deformed momentum conservation law does not pick out a preferred reference frame). From the results of [21] it looked like the κ-Poincaré model was invariant under the modified Poincaré transformations provided they were implemented in this way. However, it turns out that there remains an issue with the well-definition of boosts in the case that particles are allowed to interact more than once in their lifetimes: when a particle interacts twice, it must participate in two interaction vertices and hence it gets a backreaction from both of these vertices. These two backreactions are in general not compatible with each other and this leads to inconsistencies when multiple interactions are considered. The main result of this thesis is a proposal for implementing the boosts on interacting particles in a modified way. This proposal leads to invariance under boosts in many interacting scenarios, although there still remain scenarios in which invariance is lost. Our approach has some connections to the approach to translational invariance for multiply interacting particles proposed in [26] (see

also [27]). We focus in this thesis on the 1+1 dimensional case, but the result can be extended easily to 3+1 dimensions.
Another result of our research is that the κ-Poincaré model is not covariantly compatible with the Relative Locality framework (RLF), meaning that, although the model can be accurately described by the RLF, this formulation does not behave covariantly under all κ-Poincaré transformations. The reason is that when the model is described by the RLF there is an additional equation, relating the endpoints of particle worldlines in interactions, and this equation is not invariant under boosts. It turns out that when one replaces the κ-Poincaré composition law by another, covariant[7], composition law – the one found in [28]) – then boosts do become symmetries of the full RLF, but in that case it is the translations that pose a problem, because the translations as established in [26] are not applicable in this case.

Structure of the Thesis

We will be dealing in this thesis with a lot of concepts that are probably not familiar to a lot of physicists (notably curved momentum spaces and h-adic Hopf algebras). We start therefore in part I by introducing these concepts. Chapter 1 will be a light start, covering de Sitter space, which will in fact be familiar to most. Chapter 2 deals with the general framework of Relative Locality and curved momentum spaces, and in particular, in section 2.3 we discuss the case in which the momentum space geometry is taken to be de Sitter. Chapter 3 then introduces the theory of Hopf algebras, and in particular that of h-adic Hopf algebras, of which the κ-Poincaré Hopf algebra (section 3.5) is an example. We develop some theory about this type of Hopf algebra which is usually omitted in physics text, but in our opinion is essential to understand because it is what allows one to do certain manipulations, like calculating expressions term by term in power series, or to make use of so-called nonlinear basis transformations, generalizing the linear basis transformations that are used in the theory of Lie algebras. This then concludes part I and brings us to part II, which is all about the κ-Poincaré model. Chapter 4 introduces the free κ-Poincaré particle, chapter 5 adds to this the description of a single interaction, and in chapter 6 we identify the problem that arises when particles interact more than once, and

[7]Covariant in the conventional way, i.e., without the need of a backreaction.

we propose our solution. Chapter 7 investigates the effects of replacing the κ-Poincaré composition law by the alternative momentum composition law that was found in [28]. And, finally, in chapter 8 we conclude by summarizing our results and discussing the possibilities for future research.

Part I

Prerequisites

Chapter 1

De Sitter Space

1.1 Comoving Coordinates

De Sitter (dS) space is an analog in Lorentzian signature of what in Euclidean signature is the sphere. $N+1$ Dimensional de Sitter space of 'radius' $\kappa > 0$ is defined as the submanifold of $N+2$ dimensional Minkowski space consisting of those points $(\eta^0, \ldots, \eta^{N+1})$ that satisfy

$$-\eta_{\mu\nu}\eta^\mu\eta^\nu = -(\eta^0)^2 + \sum_{i=1}^{N+1}(\eta^i)^2 = \kappa^2, \tag{1.1}$$

endowed with the induced metric. Here $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, \ldots, -1)$ is the ambient Minkowski metric. Part of this submanifold, one might say *half* of it, may be parameterized by coordinates $(x^0, \ldots, x^N) \in \mathbb{R}^{N+1}$ via the embedding

$$\eta^0 = \kappa \sinh\left(\frac{x^0}{\kappa}\right) + \frac{|\vec{x}|^2 e^{\frac{x^0}{\kappa}}}{2\kappa}, \tag{1.2}$$

$$\eta^i = x^i e^{x^0/\kappa}, \tag{1.3}$$

$$\eta^{N+1} = \kappa \cosh\left(\frac{x^0}{\kappa}\right) - \frac{|\vec{x}|^2 e^{\frac{x^0}{\kappa}}}{2\kappa}, \tag{1.4}$$

where $i = 1, \ldots, N$ and $|\vec{x}|^2 = \sum_{i=1}^N (x^i)^2$. These so-called *comoving coordinates* cover that part of de Sitter space for which $\eta^0 + \eta^{N+1} > 0$, which is visualized in

fig. 1.1. The inverse of the embedding map is given by

$$x^0 = \kappa \ln\left(\frac{\eta^0 + \eta^{N+1}}{\kappa}\right), \qquad x^i = \frac{\kappa\,\eta^i}{\eta^0 + \eta^{N+1}}. \tag{1.5}$$

The induced metric of dS space in the comoving chart can now be found as the pullback of the Minkowski metric under the embedding map, which in coordinates is given by

$$g_{ab} = \eta_{\mu\nu}\frac{\partial\eta^\mu}{\partial x^a}\frac{\partial\eta^\nu}{\partial x^b} = \mathrm{diag}\left(1, -e^{2x^0/\kappa}, -e^{2x^0/\kappa}, \ldots, -e^{2x^0/\kappa}\right). \tag{1.6}$$

Hence the de Sitter line element in comoving coordinates is given by

$$\mathrm{d}s^2 = (\mathrm{d}x^0)^2 - e^{2x^0/\kappa}\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j. \tag{1.7}$$

1.2 Killing Vector Fields and the dS Algebra

The aim of this section is to describe the symmetries of de Sitter space in $N+1$ dimensions. We list the Killing vectors fields (KVFs) and use them to derive the ($N+1$ dimensional) de Sitter algebra $\mathfrak{so}(N+1,1)$. The (finite) isometries can then be found by integrating the KVFs. We will not explicitly do this, as we will not need it.

From the metric (1.6) the following nonvanishing Christoffel symbols of can be derived, where $H = 1/\kappa$,

$$\Gamma^0_{ii} = He^{2Ht}, \qquad \Gamma^i_{0i} = \Gamma^i_{i0} = H, \qquad i = 1, \ldots N. \qquad \text{(no summation)} \tag{1.8}$$

Killing's equation $\nabla_\mu\,\xi_\nu + \nabla_\nu\,\xi_\mu = 0$ then contains $(N+1)(N+2)/2$ independent components, given (in terms of ξ with upper indices) by

$$\partial_0\,\xi^0 = 0, \tag{1.9}$$

$$\partial_i\,\xi^0 = e^{2Hx^0}\partial_0\,\xi^i, \tag{1.10}$$

$$\partial_i\,\xi^i = -H\xi^0, \qquad \text{(no summation)} \tag{1.11}$$

$$\partial_i\xi^j = -\partial_j\xi^i. \tag{1.12}$$

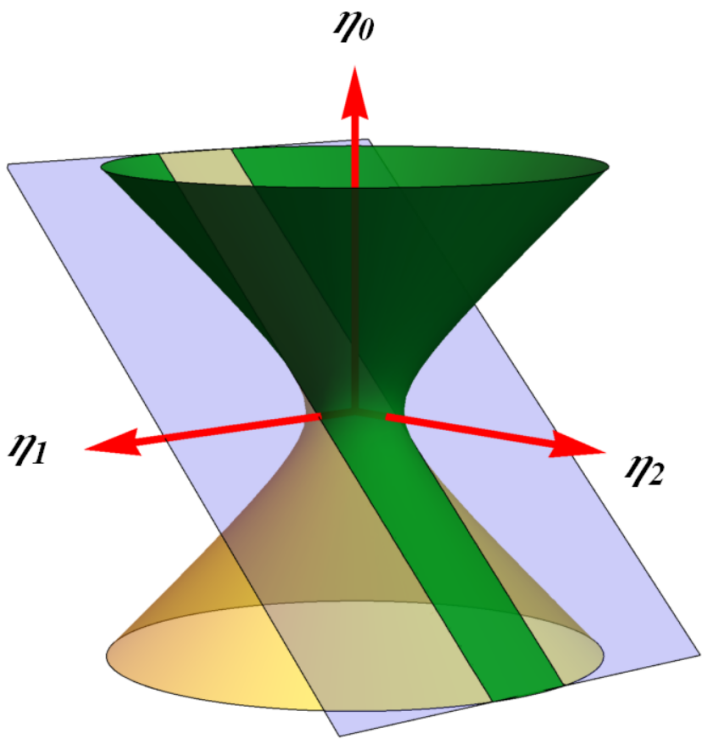


Figure 1.1: De Sitter space in 1+1 dimensions embedded as a hyperboloid in 2+1 dimensional Minkowski space. The green part of the hyperboloid, separated from the yellow part by the blue plane, is the patch of dS space that is covered by comoving coordinates.

These can be seen to have the solutions P_μ, R_{ij} and N_i for $\mu = 0, \ldots, N$, $i = 1, \ldots, N$, given by

$$P_0 = \partial_0 - \sum_i H x^i \partial_i, \qquad P_i = \partial_i, \qquad R_{ij} = x^i \partial_j - x^j \partial_i,$$
$$N_i = x^i \partial_0 - H x^i x^j \partial_j + \left(\frac{1 - e^{-2Hx^0}}{2H} + \frac{H}{2} |\vec{x}|^2 \right) \partial_i. \tag{1.13}$$

These comprise $(N+1)(N+2)/2$ linearly independent KVFs (since $R_{ij} = -R_{ji}$), so we conclude that the manifold is maximally symmetric and that any other KVF is a linear combination of the ones above. From a physics perspective, the KVFs P_μ are the generators of translations, the R_{ij} of rotations, and the N_i of boosts. In the limit $H \to 0$ they reduce to the standard translations, rotations and boost on Minkowski space. The KVFs form an $\mathfrak{so}(N+1,1)$ algebra,

$$\begin{aligned}
&[R_{ij}, R_{kl}] = -\delta_{ik} R_{jl} - \delta_{jl} R_{ik} + \delta_{il} R_{jk} + \delta_{jk} R_{il}, \qquad [R_{ij}, P_k] = -\delta_{ik} P_j + \delta_{jk} P_i, \\
&[R_{ij}, P_0] = 0, \qquad [N_i, P_j] = -\delta_{ij} P_0 + H R_{ij}, \qquad [N_i, P_0] = -P_i + H N_i, \\
&[R_{ij}, N_k] = -\delta_{ik} N_j + \delta_{jk} N_i, \\
&[N_i, N_j] = R_{ij}, \qquad [P_i, P_j] = 0, \qquad [P_0, P_i] = H P_i,
\end{aligned} \tag{1.14}$$

which can be formulated even more compactly by defining the antisymmetric matrix $M_{\mu\nu}$, where $M_{ij} = R_{ij}$ and $M_{i0} = N_i$, so that the algebra reads

$$[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\mu\rho} M_{\nu\sigma} + \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma} \tag{1.15}$$
$$[M_{\mu\nu}, P_\rho] = \eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu - H \left(\eta_{\mu 0} M_{\nu\rho} - \eta_{\nu 0} M_{\mu\rho} \right) \tag{1.16}$$
$$[P_i, P_j] = 0, \qquad [P_0, P_i] = H P_i. \tag{1.17}$$

From this we quickly see that the algebra is a deformation of the Poincaré algebra, with deformation parameter H.

1.3 dS Space in 1+1 Dimensions

Here we specialize to $1+1$ dimensions, the case with which we will be concerned the most throughout the thesis. In this case, the comoving dS metric reads

$\mathrm{d}s^2 = \mathrm{d}t^2 - e^{2Ht}\mathrm{d}x^2$ and the independent KVFs, eq. (1.13), reduce to

$$P_0 = \partial_t - Hx\,\partial_x, \tag{1.18}$$

$$P_1 = \partial_x, \tag{1.19}$$

$$N \equiv N_1 = x\,\partial_t + \left(\frac{1-e^{-2Ht}}{2H} - \frac{H}{2}x^2\right)\partial_x. \tag{1.20}$$

These comprise one (generalized) time translation, one spatial translation and one boost, respectively. Eq. (1.14) shows that these KVFs satisfy the following $\mathfrak{so}(2,1)$ Lie algebra,

$$[P_0, P_1] = HP_1, \qquad [P_0, N] = P_1 - HN, \qquad [P_1, N] = P_0. \tag{1.21}$$

The conserved charges[1] (denoted with the same symbol as the corresponding KVF) are

$$P_0 = p_0 - Hxp_1, \qquad P_1 = p_1, \qquad N = xp_0 + \left(\frac{1-e^{-2Ht}}{2H} - \frac{H}{2}x^2\right)p_1, \tag{1.22}$$

where the p components are lowered so that we obtain a cotangent bundle picture, which is suitable for a Hamiltonian description of physics. If we endow the cotangent bundle with minus(!) the canonical Poisson structure[2], so that in particular $\{x^\mu, p_\nu\} = -\delta^\mu_\nu$, then the conserved charges satisfy the same Lie algebra as the KVFs (1.21) in terms of Poisson brackets:

$$\{P_0, P_1\} = HP_1, \qquad \{P_0, N\} = P_1 - HN, \qquad \{P_1, N\} = P_0. \tag{1.23}$$

This is in fact a manifestation of a the following general result.

Proposition 1. *Let M be a smooth manifold and $\mathfrak{g} \subset \mathfrak{X}(M)$ a Lie algebra of smooth vector fields on M, under the commutator bracket. Then the map $\Pi : \mathfrak{g} \to C^\infty(T^*M)$ given by $X \mapsto \Pi_X \equiv X^\mu p_\mu$ is a Lie algebra homomorphism, where the Lie algebra structure on $C^\infty(T^*M)$ is given by minus(!) the standard Poisson bracket. We note that for the expression $X^\mu p_\mu$ to make sense, X^μ must be interpreted as a function on the cotangent bundle T^*M, as must p_μ.*

[1]The conserved charge (along a geodesic $x^\mu(\lambda)$) corresponding to some KVF ξ^μ is defined as $\Pi = \xi^\mu p_\mu$, where $p^\mu = m\dot{x}^\mu$ with m the mass of the corresponding particle and the dot representing the proper time derivative. If the mass vanishes, however, we usually set $m = 1$.

[2]Alternatively, one could use the standard Poisson structure, but change all conserved currents by an overall minus sign.

Proof. Let $X, Y \in \mathfrak{g}$. We compute

$$\{\Pi_X, \Pi_Y\} = X^\mu p_\nu \{p_\mu, Y^\nu\} + p_\mu Y^\nu \{X^\mu, p_\nu\} = X^\mu p_\nu \frac{\partial Y^\nu}{\partial x^\mu} - p_\mu Y^\nu \frac{\partial X^\mu}{\partial x^\nu} \quad (1.24)$$

$$= \left(X^\nu \frac{\partial Y^\mu}{\partial x^\nu} - Y^\nu \frac{\partial X^\mu}{\partial x^\nu} \right) p_\mu = [X, Y]^\mu p_\mu = \Pi_{[X,Y]}. \quad (1.25)$$

In the first step we have used the definition of the map Π and the derivation property of the Poisson bracket. In the second step we have used the explicit form of the Poisson bracket,

$$\{f, g\} = \frac{\partial f}{\partial p_\mu} \frac{\partial g}{\partial x^\mu} - \frac{\partial f}{\partial x^\mu} \frac{\partial g}{\partial p_\mu}. \quad (1.26)$$

And the third step can easily be seen to be true by acting with the commutator of two vector fields on a function. □

This result has the following useful corollary.

Corollary 2. *The map that sends the vector field X on M to the Hamiltonian vector field $\{\Pi_X, \cdot\}$ on T^*M is a Lie algebra homomorphism.*

Proof. The map $f \mapsto \{f, \cdot\}$ is a Lie algebra homomorphism from the smooth functions to smooth vector fields, hence the resulting map $X \mapsto \{\Pi_X, \cdot\}$ is a composition of Lie algebra homomorphisms. □

And a similar result in this spirit, which will also be useful later, is the following one.

Proposition 3. *Let M be a manifold and X some vector field with flow $\phi_t^X(x)$. Write $\Pi_X = X^\mu p_\mu \in C^\infty(T^*M)$, and let $\pi : T^*M \to M$ be the canonical projection. Let $\phi^{\{\Pi_X, \cdot\}}$ be the flow of the Hamiltonian vector field corresponding to Π_X. Then we have*

$$\pi \circ \phi^{\{\Pi_X, \cdot\}} = \phi^X. \quad (1.27)$$

This means that the vector field X and the vector field $\{\Pi_X, \cdot\}$ generate the same curve on the base manifold.

Proof. The Hamiltonian (vector field) flow is determined by the (Hamilton's) equations

$$\dot{x}^\mu = \{x^\mu, \Pi_X\} = \{x^\mu, X^\nu(x)p_\nu\} = X^\nu(x)\{x^\mu, p_\nu\} = X^\nu \delta^\mu_\nu = X^\mu, \tag{1.28}$$

$$\dot{p}_\mu = \dots . \tag{1.29}$$

The first equation is precisely the defining equation for the flow of X itself, and the second equation is merely a definition of what is meant by p_μ, hence irrelevant for the trajectory on the base manifold. □

1.3.1 Geodesics

With the Christoffel symbols (1.8) reducing to

$$\Gamma^0_{11} = He^{2Ht}, \qquad \Gamma^1_{01} = \Gamma^1_{10} = H, \tag{1.30}$$

we find that the geodesic equation, $\ddot{x}^\rho + \Gamma^\rho_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$, contains the following component equations

$$\ddot{t} = -He^{2Ht}\dot{x}^2, \tag{1.31}$$

$$\ddot{x} = -2H\dot{x}\dot{t}. \tag{1.32}$$

To solve the system of equations, we make use of the fact that the conserved charge $P_1 = p_1$ is constant along geodesics. Let us assume for the moment that $m \neq 0$. Then the conserved charges read

$$\begin{aligned} P_1 = p_1 = mg_{1\mu}\dot{x}^\mu = -me^{2Ht}\dot{x}^1 = -me^{2Ht}\dot{x}, \\ \Rightarrow \qquad \dot{x} = -\frac{p_1}{m}e^{-2Ht}. \end{aligned} \tag{1.33}$$

Plugging this into (1.31) leads to

$$\ddot{t} = -\left(\frac{p_1}{m}\right)^2 He^{-2Ht} \tag{1.34}$$

and from this it follows, using

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\dot{t}^2\right) = \ddot{t}\,\dot{t} = -\left(\frac{p_1}{m}\right)^2 He^{-2Ht}\,\dot{t} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\left(\frac{p_1}{m}\right)^2 e^{-2Ht}\right), \tag{1.35}$$

that

$$\dot{t} = \pm\sqrt{\left(\frac{p_1}{m}\right)^2 e^{-2Ht} + c_1}, \tag{1.36}$$

with c_1 an integration constant, which is found to be equal to 1 by imposing the normalization condition

$$1 \stackrel{!}{=} g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \dot{t}^2 - e^{2Ht}\dot{x}^2 = \left(\frac{p_1}{m}\right)^2 e^{-2Ht} + c_1 - e^{2Ht}\left(\frac{p_1}{m}e^{-2Ht}\right)^2 = c_1. \tag{1.37}$$

Not that by multiplying this equation by m^2 it also provides us with the energy-momentum dispersion relation

$$m^2 = m^2 g^{\mu\nu}p_\mu p_\nu = p_0^2 - e^{-2Ht}p_1^2. \tag{1.38}$$

We will choose the $\pm$ sign in the expression (1.36) for $\dot{t}$ to be positive, in order to have positive time orientation and positive energy. Combining these results with (1.33) yields

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\dot{x}}{\dot{t}} = \frac{-\frac{p_1}{m}e^{-2Ht}}{\sqrt{\left(\frac{p_1}{m}\right)^2 e^{-2Ht} + 1}} = \frac{-\mathrm{sign}(p_1)\, e^{-2Ht}}{\sqrt{e^{-2Ht} + \left(\frac{m}{p_1}\right)^2}}, \tag{1.39}$$

which can be integrated to

$$x(t) = \bar{x} + \frac{\mathrm{sign}(p_1)}{H}\left(\sqrt{e^{-2Ht} + \left(\frac{m}{p_1}\right)^2} - \sqrt{1 + \left(\frac{m}{p_1}\right)^2}\right) \tag{1.40}$$

with $\bar{x} = x(t = 0)$. This is the general form of a worldline of a massive particle in dS spacetime. If we set $m = 0$, the worldline reduces nicely to

$$x(t) = \bar{x} + \frac{\mathrm{sign}(p_1)}{H}\left(e^{-Ht} - 1\right), \tag{1.41}$$

Note that technically our derivation is not really valid for $m = 0$, for in that case our definition of p yields $p = 0$, and the normalization would not be correct either. In the massless case we should define $p^\mu = \dot{x}^\mu$ instead and use the normalization $p_\mu p^\mu = 0$. (The physical momentum would then be a scalar multiple of p^μ.) The result, however, would be precisely the same as what we obtain by simply setting $m = 0$ in eq. (1.40).

1.3.2 Worldines from a Symmetry Approach

There is in fact an alternative way of finding the geodesics in de Sitter space, namely by taking the symmetry algebra as the starting point. This approach can be generalized to different symmetry algebras, and even (quantum) deformations of symmetry algebras. We will find this very useful later when dealing with the κ-Poincaré Hopf algebra, so we will here illustrate the procedure, reproducing the results of section 1.3.1.

In a nutshell, the idea is that we represent a given Lie algebra on phase space in terms of Poisson brackets, pick a Casimir element, which we require to be equal to the mass-squared of the particle considered, and choose our Hamiltonian to be proportional to the same Casimir element. This nicely implements the fact that the Lie algebra should correspond to symmetries: all elements of the (represented) Lie algebra are symmetries if and only if they all Poisson-commute with the Hamiltonian; and in particular the mass should be an invariant. If the Lie algebra has more than one Casimir (modulo scaling) then the choice of Hamiltonian is, of course, ambiguous in this setting. However, a standard choice can always be made whenever the Killing form of the Lie algebra is nondegenerate: the quadratic Casimir induced by the Killing form.
In the present case of $1+1$ dS space, the Lie algebra is $\mathfrak{so}(2,1)$, eq. (1.21) and, by prop. 1, we already have a natural representation of the algebra on phase space, namely the one consisting of the conserved charges (1.22) of the Killing vector fields. The quadratic Casimir[3] of the algebra is

$$C = (P_0)^2 - (P_1)^2 + 2HNP_1. \tag{1.42}$$

This Casimir will play a double role. First we will require it, for each given particle, to be equal to its mass-squared. Second, it will be (proportional to) our Hamiltonian, generating the evolution in an affine parameter λ. Using our phase space representation (1.22) of the algebra, the Casimir reduces to

$$C = (p_0)^2 - e^{-2Hx^0}(p_1)^2 = m^2. \tag{1.43}$$

[3]The quadratic Casimir corresponding to the Killing form defined as $K(X,Y) = \mathrm{Tr}(\mathrm{ad}(X)\mathrm{ad}(Y))$ has in fact an extra multiplicative factor $1/2H^2$ in front of it. We do not include it.

Already we see that this dispersion relation is identical to the one we obtained in the previous section using General Relativity, eq. (1.38). Note again that we need to take *minus* the standard Poisson structure on phase space for our representation to actually be a representation. We must now split the analysis into two parts: massive particles and massless particles. They require a slightly different Hamiltonian.

Massive Particles

As mentioned above, we will require that $C = m^2$, but as Hamiltonian we cannot just take C, because of dimensional consistency[4]. Therefore we take[5] $H = C/2m$, the m factor being there to make all dimensions match, and the factor 2 being there in order that the canonical p_0, p_1 used here will correspond to the (physical) momentum, i.e., the one from the General Relativity approach of the previous section. Hamilton's equations read

$$\dot{x}^0 = \{x^0, H\} = \frac{p_0}{m}, \tag{1.44}$$

$$\dot{x}^1 = \{x^1, H\} = -\frac{p_1}{m} e^{-2Hx^0}, \tag{1.45}$$

$$\dot{p}_0 = \{p_0, H\} = -\frac{H(p_1)^2}{m} e^{-2Hx^0}, \tag{1.46}$$

$$\dot{p}_1 = \{p_1, H\} = 0. \tag{1.47}$$

The last equation shows that p_1 must be a constant. Using the dispersion relation, we may rewrite the first two equations as

$$m\dot{x}^0 = \pm\sqrt{e^{-2Hx_0}(p_1)^2 + m^2}, \tag{1.48}$$

$$m\dot{x}^1 = -p_1 e^{-2Hx_0}, \tag{1.49}$$

[4]If we took just C as our Hamiltonian, then Hamilton's equations on the one hand and the dispersion relation on the other hand would each require the canonical momentum p_μ to have a different dimension, contradicting each other.

[5]Instead of taking this Hamiltonian and requiring $C = m^2$, we might take the Hamiltonian $C/2m - m/2$, requiring that this Hamiltonian vanishes on physical (i.e. on-shell) trajectories.

where, as before, we choose $\pm = +$ in order that we have positive time orientation and positive energy. Thus the quotient $\dot{x}^1/\dot{x}^0$ satisfies (assuming $p_1 \neq 0$)

$$\frac{\dot{x}^1}{\dot{x}^0} = \frac{-p_1 e^{-2Hx_0}}{\sqrt{e^{-2Hx_0}(p_1)^2 + m^2}} = \frac{-\text{sign}(p_1)\, e^{-2Hx_0}}{\sqrt{e^{-2Hx_0} + \left(\frac{m}{p_1}\right)^2}}. \tag{1.50}$$

This is exactly the same equation as (1.39), so the projected worldlines in this case must also be given by (1.40). It can be checked that with this solution all of Hamilton's equations are satisfied and the dispersion relation as well. The relation between p_μ and x^μ is also the same as in the former analysis. That means that the canonical momenta p_μ used here are simply the (index-lowered versions of the) physical momenta. Hence we have found that the physics described by this Hamiltonian setup is identical to that described in the previous section using the framework of General Relativity. As already mentioned earlier there are some ambiguities in this framework: 1) Why did we use the specific Casimir C to represent the mass of the particle? 2) Why did we use $C/2m$ as Hamiltonian? The factor $1/m$ was necessary from dimensional analysis, but the numerical factor $1/2$ could have been anything, *a priori*. Note however that rescaling the Hamiltonian by a numerical factor does not affect the (projected) worldlines, so the latter ambiguity amounts in fact to nothing more than a mathematical convention. And in $1+1$ dimensions there is only one Casimir (up to scaling) in the dS algebra, so in this particular case there are in fact no ambiguities at all. Actually, even in $1+3$ dimensions, although there are two linearly independent Casimir elements in the dS algebra, only one of those will correspond to the Special Relativistic energy-momentum dispersion relation in the flat limit, so in that case there is no ambiguity either. Nevertheless, these are all particular cases and in the most general case the ambiguity of picking a Casimir element will remain.

Massless Particles

For massless particles we cannot divide by m, so a different Hamiltonian is needed. In this case, however, we *can* use the simpler choice of Hamiltonian[6],

[6]In this case we can formulate the setup alternatively as being that of an Hamiltonian $H = C/2$ generating the evolution in an affine parameter, with the additional requirement

$H = C/2$. We are not plagued by dimensional inconsistencies now, because the equation $C = m^2$ can in this case simply be reduced to $C = 0$. This equation does *not* require that C must have the dimension of mass-squared, and therefore it does not contradict Hamilton's equations. Hamilton's equations read

$$\dot{x}^0 = \{x^0, H\} = p_0, \tag{1.51}$$
$$\dot{x}^1 = \{x^1, H\} = -p_1 e^{-2Hx^0}, \tag{1.52}$$
$$\dot{p}_0 = \{p_0, H\} = -H(p_1)^2 e^{-2Hx^0}, \tag{1.53}$$
$$\dot{p}_1 = \{p_1, H\} = 0. \tag{1.54}$$

Using the dispersion relation, we may rewrite the first two equations as

$$\dot{x}^0 = e^{-Hx_0}|p_1|, \tag{1.55}$$
$$\dot{x}^1 = -p_1 e^{-2Hx_0}, \tag{1.56}$$

where we have again chosen positive time orientation. Thus the quotient $\dot{x}^1/\dot{x}^0$ satisfies (assuming $p_1 \neq 0$)

$$\frac{\dot{x}^1}{\dot{x}^0} = \frac{-p_1 e^{-2Hx_0}}{e^{-Hx_0}|p_1|} = -\text{sign}(p_1)\, e^{-Hx_0}. \tag{1.57}$$

This is again the same equation as (1.39), with $m = 0$. Hence the worldlines of massless particles, in this approach, are also the same ones as those obtained by the conventional general relativity approach.

The preceding results bear the question *why* the two completely different approaches, section 1.3.1 versus section 1.3.2, yield the same physics. Appendix A contains a discussion on this matter.

1.3.3 Redshift

With the relevant formulas for the worldlines of particles at hand, we are now in the position to go a bit more into the actual physics in de Sitter spacetime. In this section we demonstrate the redshift effect. The analysis will be structured

that $H = 0$ for physical (i.e. on-shell) particles.

in such a way that we can repeat a similar analysis for de Sitter *momentum space* later. This will allows us in section 4.4.1 to notice an interesting duality between these two settings.

We make use of the Hamiltonian framework as in section 1.3.2. With the phase space representation of the KVFs (eq. (1.22)), we can derive the action of the symmetries on phase space. Let us recall how this works. Consider a translation with translation parameter a^μ. With P_μ the generators of translations, the symmetry generator will be $a^\mu P_\mu$, which determines a Hamiltonian vector field $\{\rho(a^\mu P_\mu), \cdot\}$ on the cotangent bundle, where ρ is the representation, which we will omit from here onwards. This vector field generates a flow, defined by the equations

$$\dot{x}^\mu = \{x^\mu, a^\nu P_\nu\}, \qquad \dot{p}_\mu = \{p_\mu, a^\nu P_\nu\}. \tag{1.58}$$

These are just Hamilton's equations with the Hamiltonian replaced by the appropriate symmetry generator. The solutions to these equations are the integral curves to the vector field corresponding to the symmetry, that is, they are the corresponding finite symmetries. In the case of translations, we obtain

$$\dot{x}^0 = \{x^0, a^0(p_0 - Hx^1p_1) + a^1p_1\} = a^0\{x^0, p_0\} = -a^0, \tag{1.59}$$
$$\dot{x}^1 = \{x^1, a^0(p_0 - Hx^1p_1) + a^1p_1\} = a^0Hx^1 - a^1, \tag{1.60}$$
$$\dot{p}_0 = \{p_0, a^0(p_0 - Hx^1p_1) + a^1p_1\} = 0, \tag{1.61}$$
$$\dot{p}_1 = \{p_1, a^0(p_0 - Hx^1p_1) + a^1p_1\} = -a^0Hp_1. \tag{1.62}$$

Solving the equations is a simple exercise and yields

$$x^0(\lambda) = \bar{x}^0 - a^0\lambda, \tag{1.63}$$
$$x^1(\lambda) = e^{a^0H\lambda}\left(\bar{x}^1 + \frac{a^1}{a^0H}\left(e^{-a^0H\lambda} - 1\right)\right), \tag{1.64}$$
$$p_0(\lambda) = \bar{p}_0, \tag{1.65}$$
$$p_1(\lambda) = \bar{p}_1e^{-a^0H\lambda}, \tag{1.66}$$

with the barred symbols representing their value at $\lambda = 0$. The result of the symmetry transformation is now obtained by setting $\lambda = 1$. Thus we see that a

translation with parameter a^μ is given by

$$x^0 \to x^0 - a^0, \tag{1.67}$$

$$x^1 \to e^{a^0 H}\left(x^1 + \frac{a^1}{a^0 H}\left(e^{-a^0 H} - 1\right)\right), \tag{1.68}$$

$$p_0 \to p_0, \tag{1.69}$$

$$p_1 \to p_1 e^{-a^0 H}. \tag{1.70}$$

Note that these reduce to the ordinary translations on 1+1 Minkowski space in the limit $H \to 0$.
We now consider two observers, Alice and Bob. Alice will emit two photons with the same energy from her spatial origin. One at $x^0 = 0$ and the other at some other time $x^0 = \tilde{x}^0$ on her clock. Bob is defined by an a^μ-translation with respect to Alice[7] such that the first photon crosses Bobs spacetime (ST) origin. Then the second one will automatically cross his spatial origin as well. We are interested in the relation between the two energies measured by Bob when each of the particles crosses his spatial origin.

Alice will see the first photon move along the worldline

$$x^1_A = \frac{1}{H}\left(1 - e^{-H x^0_A}\right), \tag{1.71}$$

assuming that the particle moves to the 'right'. Applying the translation to x and t, we find that Bob sees the worldline

$$x^1_B = \frac{1}{H}\left(\frac{a^1}{a^0} - 1\right)\left(1 - e^{-a^0 H}\right) + \frac{1}{H}\left(1 - e^{-H x^0_B}\right). \tag{1.72}$$

The requirement that the photon crosses Bob's ST origin then translates to $x^1_B(x^0_B = 0) = 0$, or equivalently,

$$a^1 = a^0. \tag{1.73}$$

[7]This means *by definition* that all x^μ, p_μ coordinates that Bob sees are those that are found by translating by a^μ the coordinates that Alice sees, via the translations derived above.

Suppose that Alice emits the first photon with an energy[8] $p_0^{A@A}$ when her clock ticks $x^0_{A@A} = 0$. Here we use the label $A@A$ to denote that the quantity is evaluated by Alice, "A", when the photon is at Alice's spatial origin, "@A". Using the dispersion relation (1.43) we can write

$$p_0^A = p_1^A e^{-Hx_A^0} = p_0^{A@A} e^{-H\left(x_A^0 - x^0_{A@A}\right)}, \tag{1.74}$$

so that from Alice's point of view the photon will have an energy

$$p_0^{A@B} = p_0^{A@A} e^{-H\left(x^0_{A@B} - x^0_{A@A}\right)}, \tag{1.75}$$

when it crosses Bob's spatial origin, which has coordinate $x^0_{A@B} = a^0$. In order that we can use the formulas again for the second photon, we explicitly keep the $x^0_{A@A}$-dependence, despite the fact that $x^0_{A@A} = 0$ for this photon. Since p^0 is invariant under under translations, Bob also measures this energy:

$$p_0^{B@B} = p_0^{A@A} e^{-H\left(x^0_{A@B} - x^0_{A@A}\right)}. \tag{1.76}$$

The second photon

Now suppose Alice emits a second photon, with the same energy $\tilde{p}_0^{A@A} = p_0^{A@A}$, and in the same direction. She does this when her clock reads $\tilde{x}_A^0$. Since we kept the dependence on x_A^0 in our formulas for the first photon, we immediately know the formula for the second photon:

$$\tilde{p}_0^{B@B} = \tilde{p}_0^{A@A} e^{-H\left(\tilde{x}^0_{A@B} - \tilde{x}^0_{A@A}\right)} = p_0^{A@A} e^{-H\left(\tilde{x}^0_{A@B} - \tilde{x}^0_{A@A}\right)}. \tag{1.77}$$

The relation between the two energies measured by Bob is therefore given by

$$\tilde{p}_0^{B@B} = e^{-H\left(\Delta_A \tilde{x}^0 - \Delta_A x^0\right)} p_0^{B@B}. \tag{1.78}$$

where we have defined the travel times according to Alice

$$\Delta_A x^0 \coloneqq x^0_{A@B} - x^0_{A@A}, \qquad \Delta_A \tilde{x}^0 \coloneqq \tilde{x}^0_{A@B} - \tilde{x}^0_{A@A}. \tag{1.79}$$

We will see in section 4.4.1 that if, instead of spacetime, momentum space has de Sitter geometry, there is an analogous scenario in which we obtain a similar equation, eq. (4.55), where Δx and p are interchanged with respect to the present equation (1.78), exposing an interesting duality between the two models.

[8]Note that the p^μ in this analysis actually represent the canonical momenta. The physical energy and momentum are a scalar multiple of these.

1.4 dS Space in 3+1 Dimensions

In this section we will generalize some of the $1+1$ dimensional results to $3+1$ dimensions. In this case one usually defines the rotation about the ith axis as $R_i = -\frac{1}{2}\epsilon_{ijk}R_{jk}$ in terms of the rotations (1.13) and the completely antisymmetric symbol ϵ_{ijk} with $\epsilon_{123} = 1$. The algebra (1.14) then becomes

$$\begin{aligned}
&[P_0, P_i] = HP_i, && [P_0, N_i] = P_i - HN_i, && [P_0, R_i] = 0,\\
&[P_i, P_j] = 0, && [P_i, N_j] = \delta_{ij}P_0 - H\epsilon_{ijk}R_k, && [P_i, R_j] = \epsilon_{ijk}P_k,\\
&[N_i, N_j] = -\epsilon_{ijk}R_k, && [R_i, R_j] = \epsilon_{ijk}R_k, && [N_i, R_j] = \epsilon_{ijk}N_k.
\end{aligned} \tag{1.80}$$

The conserved charges corresponding to the KVFs (which we also denote simply by P_0) are

$$P_0 = p_0 - Hx^1p_1 - Hx^2p_2 - Hx^3p_3, \qquad P_1 = p_1, \qquad P_2 = p_2, \qquad P_3 = p_3, \tag{1.81}$$

$$R_1 = x^3p_2 - x^2p_3, \tag{1.82}$$

$$R_2 = -x^3p_1 + x^1p_3, \tag{1.83}$$

$$R_3 = x^2p_1 - x^1p_2, \tag{1.84}$$

$$N_1 = x^1p_0 + \left[\frac{1-e^{-2Hx^0}}{2H} - \frac{H}{2}\left((x^1)^2 - (x^2)^2 - (x^3)^2\right)\right]p_1 - Hx^1x^2p_2 - Hx^1x^3p_3, \tag{1.85}$$

$$N_2 = x^2p_0 + \left[\frac{1-e^{-2Hx^0}}{2H} - \frac{H}{2}\left((x^2)^2 - (x^1)^2 - (x^3)^2\right)\right]p_2 - Hx^1x^2p_1 - Hx^2x^3p_3, \tag{1.86}$$

$$N_3 = x^3p_0 + \left[\frac{1-e^{-2Hx^0}}{2H} - \frac{H}{2}\left((x^3)^2 - (x^1)^2 - (x^2)^2\right)\right]p_3 - Hx^1x^3p_1 - Hx^2x^3p_2. \tag{1.87}$$

As in the $1+1$ dimensional case, if we now endow the cotangent bundle with minus(!) the canonical Poisson structure, so that in particular $\{x^\mu, p_\nu\} = -\delta^\mu_\nu$,

then the conserved charges satisfy the same Lie algebra (in terms of Poisson brackets) as the KVFs, (1.80), which is guaranteed by prop. 1 . So again we immediately have a representation of the KVF Lie algebra on phase space.

The quadratic Casimir[9] can be calculated to be

$$C = (P_0)^2 - \vec{P}^2 + H\left(\vec{P}\cdot\vec{N} + \vec{N}\cdot\vec{P}\right) - H^2\vec{R}^2 \tag{1.88}$$

and in our phase space representation this Casimir becomes

$$C = (p_0)^2 - e^{-2Hx^0}\left((p_1)^2 + (p_2)^2 + (p_3)^2\right). \tag{1.89}$$

Notice that we can write this as $C = g^{\mu\nu}p_\mu p_\nu$.

1.4.1 Geodesics

To find the geodesics in $1+3$ dimensional de Sitter space, we will focus first only on worldlines that cross the spacetime origin. Those are easily found. Hamilton's equations are

$$\dot{x}^\mu = p^\mu, \qquad \dot{p}_0 = -He^{-2Hx^0}\left((p_1)^2 + (p_2)^2 + (p_3)^2\right), \qquad \dot{p}_i = 0. \tag{1.90}$$

The first thing to notice is that if $p_2 = p_3 = 0$ the equations reduce to the $1+1$ dimensional ones. Hence the solutions we found in $1+1$ dimensions are here solutions as well, provided we set $p_2 = p_3 = 0$. We can find the other ones using the symmetries of the metric. All we have to do is take a $1+1$ dimensional solution, rotate it, and then translate it (i.e., evaluate it in coordinates which are

[9] Actually, if we would define the Killing form as $K(X,Y) = \mathrm{Tr}(\mathrm{ad}(X)\mathrm{ad}(Y))$ then our C would have an extra factor $1/6H^2$. We will not include this factor.

rotated and then translated). Rotations act trivially:

$$R(\theta,\phi,\psi)x^\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & 0 & \sin\phi \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\phi & 0 & \cos\phi \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\psi & \sin\psi \\ 0 & 0 & -\sin\psi & \cos\psi \end{pmatrix}\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{1.91}$$

$$R(\theta,\phi,\psi)p_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & 0 & \sin\phi \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\phi & 0 & \cos\phi \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\psi & \sin\psi \\ 0 & 0 & -\sin\psi & \cos\psi \end{pmatrix}\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} \tag{1.92}$$

and a translation with parameters a^μ acts as follows,

$$x^0 \to x^0 - a^0, \qquad p_0 \to p_0, \tag{1.93}$$

$$x^i \to \left(x^i - \frac{a^i}{a^0 H}\right)e^{a^0 H} + \frac{a^i}{a^0 H}, \qquad p_i \to p_i e^{-a^0 H}, \tag{1.94}$$

which can be found by integrating the Poisson action of the translation generators. The thus obtained worldlines are given by

$$x^i(x^0) = \bar{x}^i + \frac{p_i}{H|\vec{p}|^2}\left(\sqrt{|\vec{p}|^2 e^{2a^0 H}+1} - \sqrt{|\vec{p}|^2 e^{-2x^0 H}+1}\right), \quad \bar{x}^i = \frac{a^i(1-e^{a^0 H})}{a^0 H}, \tag{1.95}$$

where $\bar{x}^i = x^i(x^0 = -a^0)$ and $|\vec{p}|^2 = \sum (p_i)^2$. Actually the leftmost plus sign in the worldline can also be chosen to be a minus, but that choice corresponds to $\dot{x}^0$ being negative, resulting in curves with incorrect time orientation and negative energies and such; we disregard those cases. We can immediately see that all geodesics must be of this form. Because for any $\bar{x}^i$ and any $a^0 \neq 0$ there exist a^i such that $\bar{x}^i = x^i(x^0 = -a^0)$, meaning that the corresponding geodesic

crosses the point $(-a^0, \bar{x}^1, \bar{x}^2, \bar{x}^3)$. If we also choose the momentum (p_0, p_1) of the 1+1 dimensional solution and rotation angles conveniently, we may obtain any possible momentum p_μ (in particular at that point), leading to any possible $p^\mu = \dot{x}^\mu$, so that the uniqueness of geodesics guarantees that we have found all possible worldlines.:

$$\boxed{x^i(x^0) = \bar{x}^i + \frac{p_i}{H|\vec{p}|^2}\left(\sqrt{|\vec{p}|^2 e^{-2a^0 H} + 1} - \sqrt{|\vec{p}|^2 e^{-2x^0 H} + 1}\right), \quad \bar{x}^i = x^i(x^0 = a^0).} \tag{1.96}$$

Geodesic Distance to the Origin

The geodesic distance from a point x^μ to the origin (the origin is defined simply as the point which has vanishing comoving coordinates) can now be computed by parameterizing a geodesic from 0 to x^μ as

$$x^\mu(t) = \left(t, x^1(t), x^2(t), x^3(t)\right). \tag{1.97}$$

in terms of (1.96) with $\bar{x}^i = 0$. It follows that

$$\dot{x}^0 = 1, \quad \dot{x}^i = \frac{p_i\, e^{-2tH}}{\sqrt{|\vec{p}|^2 e^{-2tH} + 1}}, \tag{1.98}$$

so for points in dS space that are separated from the origin by a timelike geodesic (i.e., $g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu > 0$) we can calculate

$$\begin{aligned} d(x^\mu, 0) &= \int_0^{x^0} \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}\, \mathrm{d}t = \int_0^{x^0} \sqrt{1 - \frac{|\vec{p}|^2 e^{-2tH}}{|\vec{p}|^2 e^{-2tH} + 1}}\, \mathrm{d}t \\ &= \int_0^{x^0} \frac{1}{\sqrt{|\vec{p}|^2 e^{-2tH} + 1}}\, \mathrm{d}t = \frac{1}{H} \ln\left(\sqrt{p^2 + e^{2Ht}} + e^{Ht}\right)\Big|_0^{x_0} \\ &= \frac{1}{H} \ln\left(\frac{\sqrt{p^2 + e^{2Ht}} + e^{Ht}}{\sqrt{p^2 + 1} + 1}\right). \end{aligned} \tag{1.99}$$

This expression still contains p, which we would like to eliminate in favor of x^μ. Something that comes to mind is the fact that the worldline $x^i(t)$ (with $\bar{x}^i = 0$)

depends on $\vec{p}$, and that for a given t, the map $\vec{p} \mapsto \vec{x}(t)$ is injective and hence invertible on the correct domains. That means we can write $\vec{p} = \vec{p}(t, \vec{x})$ as a function of the point $(t, \vec{x})$ that the geodesic crosses and for which we want to calculate the distance to the origin. We can then substitute $|\vec{p}|^2 \to |\vec{p}(t, \vec{x})|^2$ and that will yield $d(x^\mu, 0)$ entirely in terms of x^μ, which is our aim. Here we state the results. The map $(t, x) \mapsto |\vec{p}|^2$ is given by

$$|\vec{p}|^2(t, \vec{x}) = \frac{4H^2|\vec{x}|^2 e^{4Ht}}{e^{4Ht}\left(H^2|\vec{x}|^2 - 1\right)^2 - 2e^{2Ht}\left(H^2|\vec{x}|^2 + 1\right) + 1} \tag{1.100}$$

and this leads to the geodesic distance

$$d(x^\mu, 0) = \frac{1}{H} \ln \left(\frac{\sqrt{\dfrac{16H^4|\vec{x}|^4 e^{8Ht}}{\left(e^{4Ht}\left(H^2|\vec{x}|^2 - 1\right)^2 - 2e^{2Ht}\left(H^2|\vec{x}|^2 + 1\right) + 1\right)^2}} + e^{2Ht} + e^{Ht}}{\sqrt{\dfrac{16H^4|\vec{x}|^4 e^{8Ht}}{\left(e^{4Ht}\left(H^2|\vec{x}|^2 - 1\right)^2 - 2e^{2Ht}\left(H^2|\vec{x}|^2 + 1\right) + 1\right)^2}} + 1 + 1} \right). \tag{1.101}$$

As stated earlier, this formula holds for all x^μ which are seperated from 0 by a *timelike* geodesic. The formula is not very appealing on an aesthetic level, but fortunately there exists another formulation, namely

$$D(x^\mu, 0) = \frac{1}{H}\text{arccosh}\left(-H^2 \eta_{\mu\nu} X^\mu X^\nu\right) = \frac{1}{H}\text{arccosh}\left(\cosh(Ht) - \frac{1}{2}H^2|\vec{x}|^2 e^{Ht}\right), \tag{1.102}$$

with X^μ the embedding coordinates of Minkowski space, $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. This is a more common formula for the geodesic distance in de Sitter space[10] and it can be derived by considering de Sitter space as a subspace of Minkowski space. Clearly it does not look anything like our formula above, but nevertheless the

[10] A different version of the formula uses arccos instead of arccosh (e.g. https://arxiv.org/pdf/hep-th/0110007.pdf). The difference between the two is, effectively, a factor i. The advantage of our version is that the distance traced out by timelike trajectories is real, in contrast to that of spacelike trajectories.

two formulas coincide for all points in dS space that are separated by a timelike geodesic from 0. This is most easily seen by substituting in $D(x^\mu, 0)$ the geodesic to the origin $x^1 = x^1(x^0)$, in terms of some suitable p_μ (which always exists because by assumption there exists a timelike geodesic to the point x^μ), and then noticing that $\cosh(Hd) = \cosh(HD)$, where we take expression (1.99) for d, not (1.101). This amounts to showing that

$$\frac{1}{|\vec{p}|^2}\left(\sqrt{|\vec{p}|^2+1}\sqrt{|\vec{p}|^2+e^{2Ht}}-e^{Ht}\right)=\frac{1}{2}\left(X+\frac{1}{X}\right) \tag{1.103}$$

with

$$X = e^{Hd} = \frac{\sqrt{|\vec{p}|^2+e^{2Ht}}+e^{Ht}}{\sqrt{|\vec{p}|^2+1}+1}. \tag{1.104}$$

In the remainder of the thesis we will only use D.

Chapter 2

Relative Locality

2.1 The Relative Locality Framework

Let us first introduce here the general concept of the relativity of locality. It expresses the idea that an interaction between particles need not necessarily be localized at a single point in spacetime, at least not for all observers. Instead, a set of interacting particles may appear to some observers as separated in spacetime. The *principle of relative locality*, introduced in [22], may be stated a little bit more specifically as follows: *Interactions between particles do not appear localized to all observers,* but *when an observer is local to an interaction (that is, one of the particles interacts in the spacetime origin of the observer) the interaction will appear localized in the observer's spacetime origin (i.e.,* all *particles in the interaction will interact in the observer's spacetime origin).* This concept is closely related to the idea of curved momentum spaces, as formalized neatly in the Relative Locality framework[1] (RLF) introduced in [22].

Instead of starting with spacetime and defining momentum space as the (co)tangent space to the given spacetime manifold, the starting point in the RLF is the specification of the *momentum space geometry.* In $N+1$ (spacetime) dimensions one

[1]The RLF is usually referred to simply as 'Relative Locality', but in order to distinguish between the more general concept of the relativity of locality and the specific framework introduced in [22] we will refer to the latter as the RLF.

must specify an $N+1$ dimensional Lorentzian momentum space manifold. Spacetime, which will always be flat in the RLF, is defined as the cotangent space to this momentum space manifold. The dynamics of a single particle (given an index I for later convenience) in a given curved momentum space is then described in the RLF by the following action,

$$S_I^{\text{free}} = \int \mathrm{d}\lambda \left(-x_I^\mu \dot{p}_\mu^I + N_I \left(D(p^I)^2 - m_I^2 \right) \right). \tag{2.1}$$

The index I is not summed over unless we explicitly write a summation sign. Here $D(p^I)$ is the geodesic distance in momentum space from p^I to the origin[2], m_I is the mass of the particle, and N_I is a Lagrange multiplier that imposes the mass-shell condition

$$D(p)^2 = m^2. \tag{2.2}$$

For a system of multiple free particles one adds their respective free actions, and in case of an interaction, say between particles with momenta $p_1, \ldots, p_n$, one adds the (boundary) term

$$S^{\text{int}} = z^\mu \mathscr{K}_\mu(p_1(\lambda_0), \ldots, p_n(\lambda_0)) \tag{2.3}$$

where λ_0 is the value of the curve parameter at which the interaction will occur, $\mathscr{K}_\mu$ are the components of a so-called conservation law, which has to be specified as well, and z^μ is a Lagrange multiplier imposing the actual conservation law $\mathscr{K}_\mu = 0$. Later we will see that the value of z^μ also has a physical interpretation, in contrast to N_I. If a momentum composition law $\oplus$ is given, a standard form for the conservation law is

$$\mathscr{K}(p_1, \ldots, p_n) = (p_1 \oplus \cdots \oplus p_m) - (p_{m+1} \oplus \cdots \oplus p_n), \tag{2.4}$$

[2]Some care is required when defining the point 0 in momentum space, since in a general curved manifold there is no such preferred point. Defining 0 by requiring it to have coordinate expression 0 is highly ambiguous because of the many possible coordinate charts. A better way of defining 0 is possible in case the coordinate functions on the momentum manifold are elements in a Hopf algebra. The co-unit ϵ of this Hopf algebra then defines the coordinates of the origin via $\epsilon(P_\mu) = P_\mu(0)$. This is coordinate independent, since a change of coordinates now corresponds to a basis change of the Hopf algebra. This will become more clear in section ??.

in terms of some momentum composition rule $\oplus$, given that particles 1 to m are incoming, and the other ones outgoing. With this form for the 'conservation law' $\mathscr{K}_\mu$, the actual conservation *law* $\mathscr{K}_\mu = 0$ clearly reduces to the Special Relativistic conservation law when $\oplus = +$. It is important to remark that, concerning the interacting particles, the integral in the free action (2.1) for an *incoming* particle must run from $-\infty$ to λ_0, and for *outgoing* particles it must run from λ_0 to ∞. This yields the right boundary terms in the equations of motion and also reflects the fact that the particles come into existence or stop existing at the vertex[3]. The total action for a system of multiple particles with an interaction can then can be written as

$$\begin{aligned} S &= \sum_I S_I^{\text{free}} + S^{\text{int}} \\ &= \sum_I \pm \int_{\lambda_0}^{\pm\infty} \left(-x_I^\mu \dot{p}_\mu^I + N_I \left(D(p^I)^2 - m_I^2\right)\right) \mathrm{d}\lambda + z^\mu \mathscr{K}_\mu(p_1(\lambda_0), \ldots, p_n(\lambda_0)), \end{aligned} \tag{2.5}$$

where the upper sign is for outgoing particles and the lower sign for incoming particles. The variation of the *free* action with respect to x_I^μ yields

$$\dot{p}_\mu^I = 0, \tag{2.6}$$

implying that all momenta remain constant and reflecting the fact that, although momentum space might be curved, spacetime is just flat Minkowski space. The variation with respect to the momenta p_μ^I yields

$$\dot{x}_I^\mu = -N_I \frac{\partial C}{\partial p_\mu^I}, \qquad \text{where} \quad C(p^I) \equiv D(p^I)^2 - m_I^2. \tag{2.7}$$

Since N_I behaves identically as a multiplicative factor in each component of $\dot{x}_I^\mu$, it does not affect the worldlines of particles and its value constitutes nothing more than a convention for the normalization of the momenta.

[3]If an already existing particle interacts and still exists after the interaction, say in case of an electron emitting a photon, then in this framework the particle (the electron in the example) after the interaction is formally considered to be a different particle than before the interaction, even though it is in fact the same.

In case of an interaction, it is convenient to write the interaction term as an integral over a δ-function $\delta(\lambda - \lambda_0)$, while doing the same for the nonvanishing boundary term that results from the integration by parts of the kinetic term. Then one finds that the variation[4] of the total action with respect to p^I_μ leads to

$$0 = \dot{x}^\mu_I + N\frac{\partial C}{\partial p^I_\mu} + \left(z^\nu \frac{\partial \mathscr{K}_\nu}{\partial p^I_\mu} \pm x^\mu_I\right)\delta(\lambda - \lambda_0), \tag{2.8}$$

where the upper sign again corresponds to *outgoing* particles, and the lower sign corresponds to *incoming* particles. Hence for $\lambda \neq \lambda_0$ the δ-function does not contribute and the equations for x^μ and p_μ are identical to the ones given above for noninteracting particles. Then, by continuity, these equations must still hold at $\lambda = \lambda_0$, which means that also the factor in front of the δ-function must vanish,

$$x^\mu_I(\lambda_0) = \mp z^\nu \frac{\partial \mathscr{K}_\nu}{\partial p^I_\mu}\bigg|_{\lambda=\lambda_0}. \tag{2.9}$$

Thus we see that the value of the Lagrange multiplier z^μ is related to the space-time point where the interaction takes place for each involved particle, i.e., where the endpoint of the particles' worldlines are located. This equation, which will be called the *interaction equation* shows that the space-time point where a particle interacts will in general be different for each particle in the interaction; the particles do not interact at the same single space-time point. If $\mathscr{K}$ is symmetric under exchange of all momenta, however, we recover locality, for then the right hand side of the equation is the same for each particle. In particular, if we use the conservation law of special relativity,

$$\mathscr{K}(p_1, \ldots, p_n) = p_1 + p_2 + \cdots + p_m - p_{m+1} - \cdots - p_n, \tag{2.10}$$

we obtain $x^\mu_I(\lambda_0) = z^\mu$ for all particles, so that z^μ represents *objectively* the exact point in space-time where the particles interact. In general, locality is recovered

[4]We fix the variations of the all variables to 0 at $\pm\infty$, as usual, but it is important that we do not fix them to 0 at the (boundary) point $\lambda = \lambda_0$.

also when $|z^\mu|$ is small enough[5] (for each μ). In that case the interaction point for each particle will be small, i.e., close to zero, and hence all particles will interact approximately in the origin of the observer's coordinate system. We will see in the next section that, given some mild assumptions about translational invariance, there always exists an observer for which $z^\mu = 0$ for a given interaction, so that this observer is *local* to the interaction: locality thus has become dependent on the observer.

2.2 Translational Invariance

As we have seen, the interaction coordinate z^μ is closely related to the point(s) in spacetime where the interaction happens. One might even go so far as claiming that the z^μ have more resemblence to ordinary spacetime coordinates than the x_I^μ, as z^μ is an objective quantity corresponding to the interaction in the sense that it does not depend on the particle, whereas the x^μ coordinate where each particle interacts is in general different for each particle. At the very least we should accept that we now have two kinds of space-time points, x_I^μ and z^μ, and thus it becomes an ambiguous matter how to define a spacetime translation.

The first option, which is the traditional one, would be to define $x^\mu \to \tilde{x}^\mu = x^\mu + a^\mu$, $p_\mu \to \tilde{p}_\mu = p_\mu$, where a^μ is the translation parameter. It turns out, however, that the theory cannot be invariant under these kinds of translations unless the conservation law is the one from Special Relativity. For suppose z^μ transforms as $z^\mu \to \tilde{z}^\mu$ under such a translation, and let us impose invariance under these transformations. Because momenta, and hence also $\mathscr{K}_\mu$ and its derivatives, are invariant under translations we find

$$x_I^\mu + a^\mu = \tilde{z}^\nu M^{(I)\mu}{}_\nu \quad \Rightarrow \quad x_I^\mu = \tilde{z}^\nu M^{(I)\mu}{}_\nu - a^\mu \stackrel{!}{=} z^\nu M^{(I)\mu}{}_\nu \tag{2.11}$$

where we have written $M^{(I)\mu}{}_\nu = \pm \frac{\partial \mathscr{K}_\nu}{\partial p_\mu^I}(p^I(\lambda_0))$, with the upper (lower) sign

[5] 'Small' should be understood in the appropriate sense here, and in particular depends on the actual form of $\mathscr{K}_\mu$, but we will assume that the conservation law $\mathscr{K}$ has the dimension of momentum, like the one of spacial relativity, so that z^μ will have the dimension of length, and in this case if z^μ is sufficiently small compared to the distance resolution of the measuring apparatus, locality of the interaction is recovered.

corresponding to outgoing (incoming) particles in the interaction, and we have suppressed the value of the curve parameter. Assuming that the matrix M^I is invertible, it follows that we must have

$$\tilde{z}^\mu = z^\mu + ((M^{(I)})^{-1})^\mu{}_\nu a^\nu. \tag{2.12}$$

But for each particle I participating in the interaction this leads to a different transformation law (since each $M^{(I)}$ is in general different), whereas there is only one z^μ corresponding to the interaction. Hence the transformation of z^μ cannot be consistently chosen such that there is invariance under these translations.

There arises, however, a second option. Namely, it might be more appropriate to define a translation by $z^\mu \to \tilde{z}^\mu = z^\mu + a^\mu$, $p_\mu \to \tilde{p}_\mu = p_\mu$. Invariance of the interaction equation (2.9) then requires that

$$\tilde{x}^\mu_I = \mp(z^\nu + a^\nu)M^{(I)\mu}{}_\nu = x^\mu_I \mp a^\nu M^{(I)\mu}{}_\nu, \tag{2.13}$$

so that $x^\mu_I(\lambda_0)$ must transform as

$$\tilde{x}^\mu_I(\lambda_0) = x^\mu_I(\lambda_0) \mp a^\nu \frac{\partial \mathscr{K}_\nu}{\partial p^I_\mu}. \tag{2.14}$$

This is achieved easily by defining (now for each value of the curve parameter)

$$\tilde{x}^\mu_I = x^\mu_I \mp a^\nu \frac{\partial \mathscr{K}_\nu}{\partial p^I_\mu}, \tag{2.15}$$

where the upper (lower) sign, again, corresponds to outgoing (incoming) particles. This alternative translation was first introduced in [26], where it was also shown that the action is invariant under these translations (up to an irrelevant boundary term).

Compatibility with Multiple Interactions

The definition of translations as introduced above is a good start, but when particles are allowed to undergo multiple interactions in their lifetimes, these translations do not suffice in general to ensure translational invariance, at least

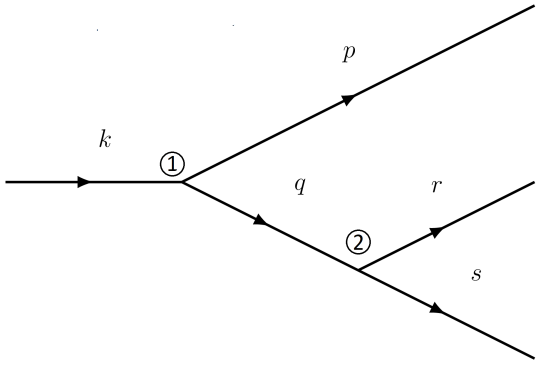


Figure 2.1: Example interaction diagram. Note that the arrows are not spinor arrows; they are only there to make the time direction clearer.

not when implemented in the naive way. In this section we show why this is the case and how to modify the implementation of the translations for the case of the standard conservation law (2.4) induced by $\oplus$ such that they are a RLF symmetry, at least for a certain class of multi-vertex diagrams. We reproduce the results of [26], where this implementation was proposed.

So first of all, let us see why the translations as just derived are not sufficient when multiple interactions are taken into account. Consider the situation as illustrated in the diagram in fig. 2.1, where time flows to the right. It is the q particle that creates the issue here, as it interacts twice. At the first vertex the momentum conservation law is $\mathscr{K}^1 = k - p \oplus q = 0$, and at the second vertex the conservation law is $\mathscr{K}^2 = q - r \oplus s = 0$. The x^μ coordinates of the q particle transform according to the first vertex as

$$\tilde{x}^\mu = x^\mu - a^\nu \frac{\partial \mathscr{K}^1_\nu}{\partial q_\mu} = x^\mu + a^\nu \frac{\partial (p \oplus q)_\nu}{\partial q_\mu}, \tag{2.16}$$

whereas according to the second vertex they transform as

$$\tilde{x}^\mu = x^\mu + a^\nu \frac{\partial \mathscr{K}^2_\nu}{\partial q_\mu} = x^\mu + a^\nu \frac{\partial q_\nu}{\partial q_\mu} = x^\mu + a^\mu. \tag{2.17}$$

There two expressions are in general not equal and hence the translations are not well-defined when a particle, like the q particle, undergoes multiple interactions. However, as was shown in [26], this can be fixed in a lot of situations. In the case at hand we would solve the problem if we could ensure that

$$\frac{\partial \mathscr{K}^2_\nu}{\partial q_\mu} = \frac{\partial (p \oplus q)_\nu}{\partial q_\mu}. \tag{2.18}$$

In order to ensure this, the conservation law $\mathscr{K}^2$ obviously needs to be modified. But the important insight is that we can change $\mathscr{K}^2$ without changing the actual conservation *law* $\mathscr{K}^2 = 0$. Instead of $\mathscr{K}^2 = q - r \oplus s$ we may use $\tilde{\mathscr{K}}^2 = p \oplus q - p \oplus (r \oplus s)$, under the assumption that $\oplus$ is a group multiplication, for in that case $\tilde{\mathscr{K}}^2 = 0$ is equivalent to $\mathscr{K}^2 = 0$. And with this modified (yet equivalent) conservation law, we indeed have

$$\frac{\partial \tilde{\mathscr{K}}^2_\nu}{\partial q_\mu} = \frac{\partial (p \oplus q)_\nu}{\partial q_\mu}, \tag{2.19}$$

so that with this form of the conservation law at vertex 2 all translations are well-defined in the given scenario. What is essential is that we have modified the conservation law at vertex 2 to be formulated in terms of the *total momentum* as computed with $\oplus$ in the 'correct' order. And we have also done this for vertex 1, but there nothing changed since the vertex already constituted the total momentum by itself.

This leads to the following generalization. When given a certain 'diagram' that shows the interactions between different particles in time, use in each vertex the conservation law in terms of the total momentum in the diagram. It is not difficult to see that this procedure yields well-defined translations for all *ordered* diagrams, which we define as diagrams that can be drawn without crossing lines (while imagining all lines extended to $\pm\infty$). In these diagrams there is an overall order of particles, namely the order as drawn from top to bottom in the diagram. And the order of particles in each vertex respects this overall order. By convention this will be the 'correct' order to add the momenta of the particles, as already alluded to before. (Note that the order is important, as $\oplus$ might be noncommutative.) That means that in these kinds of diagrams all particles have a well-defined transformation behavior under translations and the theory is

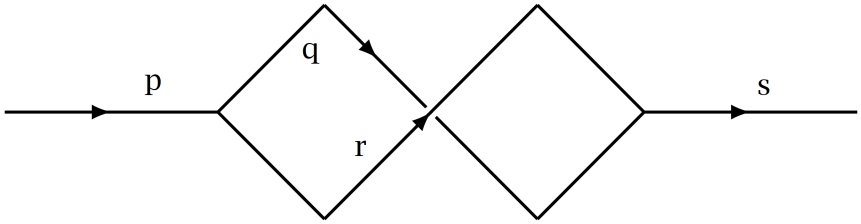


Figure 2.2: Example interaction diagram with crossing lines.

translation invariant. However, for diagrams that *do* have crossing lines one cannot guarantee that this method works, because the overall order of the particles cannot be consistent with the order in each individual vertex. A related feature of diagrams with crossing lines is that there is no *global* momentum conservation. An example of this is provided by the diagram in figure 2.2. At the first vertex, the correct order would be $q \oplus r$, while at the second vertex the correct order would be $r \oplus q$. It is easily checked that this prevents us from defining the action of translations using the method introduced above. Also, if $\oplus$ is noncommutative, global momentum conservation is lost, because $p = q \oplus r \neq r \oplus q = s$.

2.3 De Sitter Momentum Space

2.3.1 Particle Worldlines

As an explicit example we analyze the case of a 1+1 dimensional de Sitter momentum space, or rather that part of dS space which is describe by comoving coordinates. The line element in comoving coordinates (p_0, p_1), which we will identify with physical momenta, reads $\mathrm{d}s^2 = (\mathrm{d}p_0)^2 - e^{2p_0/\kappa}(\mathrm{d}p_1)^2$, where the dS radius κ is usually identified with the Planck mass. We will define the origin of this space as the point with coordinates $(0, 0)$. As derived in the previous chapter the geodesic distance to the origin is then given by

$$D(p, 0) = \kappa \operatorname{arccosh}\left(\cosh\frac{p_0}{\kappa} - e^{\frac{p_0}{\kappa}}\frac{(p_1)^2}{2\kappa^2}\right). \tag{2.20}$$

From this and the equations of motion of relative locality we find rather complicated expressions for both $\dot{x}^0$ and $\dot{x}^1$, which we omit here, but their quotient is given by

$$\frac{\partial x^1}{\partial x^0} = \frac{\dot{x}^1}{\dot{x}^0} = \frac{2\kappa p_1}{\kappa^2 \left(e^{-2p_0/\kappa} - 1\right) + (p_1)^2}. \tag{2.21}$$

Hence the worldlines in this theory are given by

$$x^1(x^0) = \bar{x}^1 + \frac{2\kappa p_1 x^0}{\kappa^2 \left(e^{-2p_0/\kappa} - 1\right) + (p_1)^2} \tag{2.22}$$

with $\bar{x}^1 = x^1(x^0 = 0)$. Interestingly, these relations are exactly the same as the ones we will obtain in the κ-Poincaré model in part 2. This can be viewed as hint that a momentum space with de Sitter geometry is somehow associated to κ-Poincaré symmetries.

2.3.2 Modified Dispersion Relation

The dispersion relation is given by the requirement that the mass of a particle is equal to its geodesic distance to the origin in momentum space. This then yields the dispersion relation

$$m = \kappa \operatorname{arccosh}\left(\cosh\frac{p_0}{\kappa} - e^{\frac{p_0}{\kappa}}\frac{(p_1)^2}{2\kappa^2}\right). \tag{2.23}$$

Notice that the mass coincides with the energy of the particle at rest ($p_1 = 0$). The dispersion relation also shows that for a given value of p_0, not all values of p_1 are allowed, because m should be real and nonnegative. By the nature of the hyperbolic cosine, this is the case if and only if inequality

$$|p_1| \leq \kappa\left(1 - e^{-p_0/\kappa}\right), \tag{2.24}$$

is satisfied, with equality when the particle is massless, $m = 0$. Note that this implies in particular that we always have $p_1 < \kappa$; the spatial momentum is bounded from above by the Planck mass.

2.3.3 Modified Momentum Conservation Law

The momentum conservation law that is often used for the a dS momentum space is the one associated to the κ-Poincaré Hopf algebra. In chapter 3 we will derive how a Hopf algebra can provide a composition law, and in chapter 5 we will see how this works in detail for the κ-Poincaré case. Here we simply state the resulting composition law,

$$(p \oplus q)_0 = p_0 + q_0, \qquad (p \oplus q)_1 = p_1 + e^{-p_0/\kappa} q_1, \tag{2.25}$$

The conservation law can then be defined by eq. 2.4. The composition has the convenient property that it respect positivity of energy and it respects the inequality 2.24. For if p and q are 'allowed' momenta, i.e., p_0 and q_0 are nonnegative and they both satisfy (2.24), then $(p \oplus q)_0$ is clearly nonnegative as well, and

$$|(p \oplus q)_1| \leq \kappa \left(1 - e^{-p_0/\kappa}\right) + e^{-p_0/\kappa} \kappa \left(1 - e^{-q_0/\kappa}\right) = \kappa \left(1 - e^{-(p_0+q_0)/\kappa}\right) \tag{2.26}$$

$$= \kappa \left(1 - e^{-(p \oplus q)_0/\kappa}\right), \tag{2.27}$$

and hence also $p \oplus q$ is an allowed momentum. In particular, the spatial component of a composition of momenta is again smaller than the deformation scale κ.

Chapter 3

Hopf Algebras and Quantum Groups

Hopf algebras are algebras with some extra structure. This extra structure is relevant in physics applications as it allows one (provided the Hopf algebra is of a certain kind) to define a multi-particle theory including energy-momentum dispersion relation and momentum composition law. The class of Hopf algebra's that we will be interested in specifically is the class of so-called h-adic Hopf algebras. Deformations of 'classical' Lie algebras are generally of this type. Before we start using Hopf algebras to specify theories of physics it is important to understand the difference between these h-adic Hopf algebras and ordinary Hopf algebras, because only the former allow us to define a multi-particle theory in the desired way. Hopf algebras are also referred to as quantum groups by some authors, but others use the term quantum group only for specific kinds of Hopf algebras. To avoid confusion we will only be using the word Hopf algebra. Standard textbooks about the subject are [29, 30, 31, 32].

3.1 Basics

To get a quick idea, a Hopf algebra is an associative unital algebra that is at the same time a coalgebra, such that some compatibility conditions are satisfied between the algebra and coalgebra sector. We will introduce these notions more

carefully below.

3.1.1 Algebras and Coalgebras

The usual definition of an algebra is as follows.

Definition 4. *An algebra (over a field $\mathbb{K}$), is a tuple (A, m), where A is a vector space (over $\mathbb{K}$) and $m : V \times V \to V$ is a bilinear map, the multiplication. We usually write $m(a, b) = a \cdot b = ab$ for all $a, b \in A$ and refer to the algebra simply as A, omitting the map m. A is called associative if $a(bc) = (ab)c$ for all $a, b, c \in A$, and A is called unital if there is exists an element $1 \in A$, called the unit, such that $a \cdot 1 = 1 \cdot a$ for all $a \in A$.*

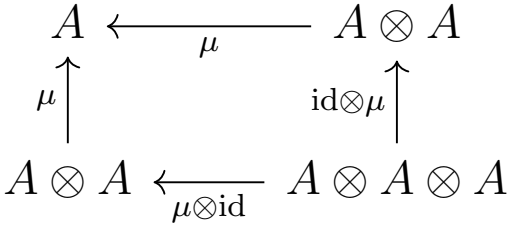
We will be interested mainly in unital associative algebras, the definition of which can be cast in a different form which will lead us naturally to the definition of coalgebras.

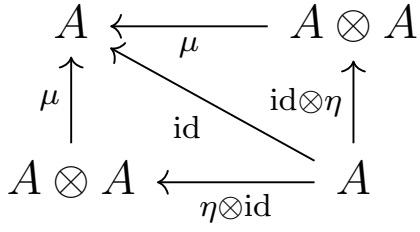
Definition 5. *A unital associative algebra over the field $\mathbb{K}$ is a triple (A, μ, η), where A is a vector space over $\mathbb{K}$, $\mu : A \otimes A \to A$ is a linear map, the multiplication, and $\eta : \mathbb{K} \to A$ is a linear map such that*

$$\mu \circ (\mu \otimes id) = \mu \circ (id \otimes \mu), \qquad \mu \circ (\eta \otimes id) = \mu \circ (id \otimes \eta) = id, \tag{3.1}$$

where id is the identity on A and in the second set of equalities the identification $\mathbb{K} \otimes A = A \otimes \mathbb{K} = A$ is made.

This says precisely that the following two diagrams are commutative.





Proposition 6. *The two given definitions of a unital associative algebra are equivalent.*

Proof. Let $(A, m, 1)$ be a unital associative algebra over $\mathbb{K}$ as in def. 4. Since $m : A \times A \to A$ is bilinear, the universal property of the tensor product ensures that there exists a unique linear map $\mu : A \otimes A \to A$ such that $\mu(a \otimes b) = m(a, b) \equiv ab$ for all $a, b \in A$. Since A is associative,

$$\mu(\mu(a \otimes b) \otimes c) = \mu(ab \otimes c) = (ab)c = a(bc) = \mu(a \otimes bc) = \mu(a \otimes \mu(b \otimes c)), \tag{3.2}$$

which, by linearity, is enough to show that $\mu \circ (\mu \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \mu)$. We now define $\eta : \mathbb{K} \to A$ as $\eta(\lambda) = \lambda \equiv \lambda \cdot 1$. Clearly η is linear, and

$$\mu(\eta(\lambda) \otimes a) = \lambda \cdot a = a \cdot \lambda = \mu(a \otimes \eta(\lambda)), \tag{3.3}$$

which, again by linearity of the involved maps, proves the first equality of $\mu \circ (\eta \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \eta) = \mathrm{id}$. The second equality is immediate by noting that $\lambda \otimes a = \lambda a$ under the canonical identification $\mathbb{K} \otimes A = A$. Thus we have shown that (A, μ, η) is a unital associative algebra in the sense def. 5. Now let us go in the opposite direction. Let (A, μ, η) be a unital associative algebra in the sense of def. 5. Define $m(a, b) = \mu(a \otimes b) \equiv ab$ and $1 = \eta(1)$. Then associativity follows the first equality in def. 5, and the fact that 1 is the unit in A follows from the last two equalities in def. 5. Hence $(A, m, 1)$ is an associative algebra in the sense of def. 4. Therefore the definitions are equivalent. □

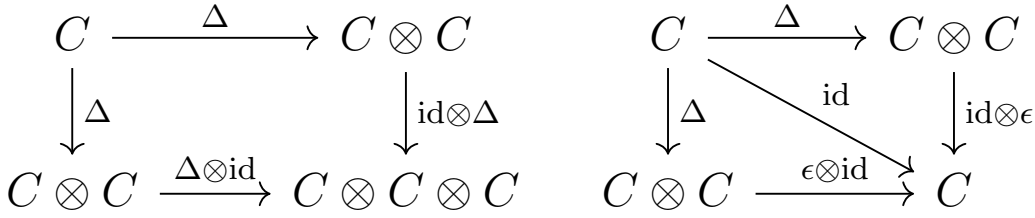
Dual to this alternative (but equivalent) definition of a unital associative algebra is the definition of a coalgebra, which is obtained by 'reversing the arrows'.

Definition 7. *A coalgebra over a field $\mathbb{K}$ is a triple (C, Δ, ϵ), where C is a vector space over $\mathbb{K}$, $\Delta : C \to C \otimes C$ is a linear map, called the coproduct, and $\epsilon : A \to \mathbb{K}$ is linear map, called the co-unit, such that*

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta, \qquad (\epsilon \otimes id) \circ \Delta = (id \otimes \epsilon) \circ \Delta = id, \tag{3.4}$$

again using the canonical identifications $\mathbb{K} \otimes A = A \otimes \mathbb{K} = A$ in the last set of equalities. The first equality expresses so-called coassociativity. A coalgebra is called cocommutative if $\Delta = \tau \circ \Delta$, where τ is the unique linear map given by $\tau(a \otimes b) = b \otimes a$.

This says that the following two 'reversed' diagrams are commutative.



A widely used and very useful notation for the coproduct is the *Sweedler notation*. Since $\Delta c \in C \otimes C$, we have $\Delta c = \sum_i c^i_{(1)} \otimes c^i_{(2)}$ for certain $c^i_{(1)}, c^i_{(2)} \in C$. In *Sweedler notation* one writes this simply as $\Delta c = c_{(1)} \otimes c_{(2)}$; the indices and the summation sign are omitted. Just like the Einstein summation convention, this can often save one a significant amount of ink.

3.1.2 Bialgebras and Hopf Algebras

We now turn to the case that a vector space is simultaneously an algebra and a coalgebra. Then there is a natural compatibility condition that may or may not be satisfied.

Definition 8. *A bialgebra is a tuple $(A, \mu, \eta, \Delta, \epsilon)$ such that (A, μ, η) is an algebra, (A, Δ, ϵ) is a coalgebra, and such that the two structures are compatible in the sense that Δ and ϵ are algebra morphisms (where $A \otimes A$ is understood to have its canonical algebra structure).*

We remark without proof or explicit definition of the following terms that said compatibility condition is equivalent to μ and η being coalgebra morphisms, justifying the use of the word 'natural' above. A Hopf algebra is now defined as follows.

Definition 9. *A Hopf algebra is a tuple $(H, \mu, \eta, \Delta, \epsilon, S)$ such that $(H, \mu, \eta, \Delta, \epsilon)$ is a bialgebra and $S : H \to H$ is a linear map, called the antipode, which satisfies*

$$\mu \circ (id \otimes S) \circ \Delta = \mu \circ (S \otimes id) \circ \Delta = \eta \circ \epsilon. \tag{3.5}$$

Let us have a look at some examples.

Example 10. *(Functions on a finite group.)*
Let G be a finite group, and define $\mathscr{F}(G) = \{\phi : G \to \mathbb{C}\}$. This function space becomes an algebra under the standard pointwise operations,

$$(\phi + \psi)(g) = \phi(g) + \psi(g), \qquad (\phi\psi)(g) = \phi(g)\psi(g). \tag{3.6}$$

In order to make $\mathscr{F}(G)$ into a coalgebra, we define the maps

$$\Delta : \mathscr{F}(G) \to \mathscr{F}(G) \otimes \mathscr{F}(G), \qquad \epsilon : \mathscr{F}(G) \to \mathbb{C}, \tag{3.7}$$

$$(\Delta\phi)(g, h) = \phi(gh), \qquad \epsilon(\phi) = \phi(1), \tag{3.8}$$

where we have identified $\mathscr{F}(G)\otimes\mathscr{F}(G) = \mathscr{F}(G\otimes G)$ via $(\phi\otimes\psi)(g, h) = \phi(g)\psi(h)$. (Note that this identification would not have been possible, in general, if G were infinite.) This makes $\mathscr{F}(G)$ not only into a coalgebra, but also a bialgebra. The antipode $S : \mathscr{F}(G) \to \mathscr{F}(G)$ given by $(S\phi)(g) = \phi(g^{-1})$ then makes $\mathscr{F}(G)$ into a Hopf algebra.

Proof. Using the Sweedler notation we have $(\Delta\otimes\mathrm{id})\Delta\phi(g, h, k) = (\Delta\otimes\mathrm{id})(\phi_{(1)}\otimes\phi_{(2)})(g, h, k) = \Delta\phi_{(1)}(g, h)\phi_{(2)}(k) = \phi_{(1)}(gh)\phi_{(2)}(k) = \Delta\phi((gh)k) = \Delta\phi(g(hk)) = \ldots = (\mathrm{id}\otimes\Delta)\Delta\phi(g, h, k)$, so that coassociativity follows directly from associativity in the group. The fact that $\mathscr{F}(G)$ is a coalgebra then follows by noticing that $((\epsilon\otimes\mathrm{id})\Delta\phi)(g) = \phi(g) = ((id\otimes\epsilon)\Delta\phi)(g)$. Next it follows easily from the fact that multiplication is pointwise that Δ and ϵ are algebra morphisms, so that $\mathscr{F}(G)$ is a bialgebra. The antipode axiom follows just as easily. □

Example 11. *(The universal enveloping algebra of a Lie algebra.)*
Consider the algebra generated by $1, T^1, \ldots, T^n$ with relations

$$\left[T^i, T^j\right] = f^{ijk}T^k, \qquad f^{ijk} \in \mathbb{R}, \tag{3.9}$$

where the brackets denote the commutator w.r.t. the algebra product. This is the universal enveloping algebra $U(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ with structure constants f^{ijk} in some basis. (The construction is independent of the chosen basis, though.) Then $U(\mathfrak{g})$ has a canonical Hopf algebra structure. Δ, ϵ and S are defined on

generators by

$$\begin{aligned} \Delta(1) &= 1 \otimes 1, & \Delta(T^i) &= 1 \otimes T^i + T^i \otimes 1, \\ \epsilon(1) &= 1, & \epsilon(T^i) &= 0, \\ S(1) &= 1, & S(T^i) &= -T^i, \end{aligned} \tag{3.10}$$

and Δ and ϵ are extended (uniquely) to algebra homomorphisms, whereas S is extended (uniquely) to an algebra antimorphism.

We omit the proof that example 11 indeed furnishes a Hopf algebra. The fact that S is extended to an algebra antimorphism in this example is related to the following result.

Proposition 12. *The antipode of any Hopf algebra is an algebra antimorphism, i.e., $S(hg) = S(g)S(h)$ for all $h, g \in H$.*

Proof. Note first that the Sweedler expression of the coassociativity axiom reads $h_{(11)} \otimes h_{(12)} \otimes h_{(2)} = h_{(1)} \otimes h_{(21)} \otimes h_{(22)}$. So also for (well-defined) functions on $H \otimes H \otimes H$, $(\mathrm{id} \otimes \epsilon \otimes \Delta)$ for instance , we may interchange arguments in this way: we may replace $h_{(11)}\epsilon(h_{(12)})\Delta(h_{(2)})$ by $h_{(1)}\epsilon(h_{(21)})\Delta(h_{(22)})$. We will employ several of those replacements in the following sequence of equalities.We will also employ the Sweedler expressions for all the other Hopf algebra axioms (which the reader can easily derive).

$$\begin{aligned} S(hg) &= S(h_{(1)}\epsilon(h_{(2)})g) = S(h_{(1)}g)\epsilon(h_{(2)}) = S(h_{(1)}g)h_{(21)}S(h_{(22)}) \\ &= S(h_{(1)}g)h_{(21)}S(h_{(22)}) = S(h_{(1)}g_{(1)}\epsilon(g_{(2)}))h_{(21)}S(h_{(22)}) \\ &= S(h_{(1)}g_{(1)})h_{(21)}\epsilon(g_{(2)})S(h_{(22)}) = S(h_{(1)}g_{(1)})h_{(21)}g_{(21)}S(g_{(22)})S(h_{(22)}) \\ &= S(h_{(11)}g_{(1)})h_{(12)}g_{(21)}S(g_{(22)})S(h_{(2)}) = S(h_{(11)}g_{(11)})h_{(12)}g_{(12)}S(g_{(2)})S(h_{(2)}) \\ &= S((hg)_{(11)})(hg)_{(12)}S(g_{(2)})S(h_{(2)}) = \epsilon((hg)_{(1)})S(g_{(2)})S(h_{(2)}) \\ &= \epsilon(h_{(1)}g_{(1)})S(g_{(2)})S(h_{(2)}) = \epsilon(h_{(1)})\epsilon(g_{(1)})S(g_{(2)})S(h_{(2)}) \\ &= S(\epsilon(g_{(1)})g_{(2)})S(\epsilon(h_{(1)})h_{(2)}) = S(g)S(h). \end{aligned} \tag{3.11}$$

□

3.2 h-adic (Hopf) Algebras

The Hopf algebras that we will be concerned with in this thesis are so-called h-adic Hopf algebras. Strictly speaking an h-adic Hopf algebra is not a Hopf algebra in the usual sense of the word, which is important to realize. Nevertheless a lot of results for ordinary Hopf algebras carry over to the h-adic case. In this section we introduce this kind of Hopf algebra and discuss some basic results that are necessary in order to do physics with them. An important aspect of h-adic Hopf algebras is that they allow for a generalization of Lie algebras, where the commutator between generators need not be a linear combination of generators but is allowed to be a nonlinear expression in the generators. More precisely, it is allowed to be a power series in some indeterminate variable h (which, in part II, will be identified with the Planck scale) with coefficients in the vector space of the corresponding 'classical' Lie algebra. For this reason we begin this section by introducing the ring of power series.

3.2.1 (Hopf) Algebras over $\mathbb{C}[[h]]$

For completeness, we begin by recalling the general definition of a ring, which is a slightly weaker notion than a field like $\mathbb{R}$ or $\mathbb{C}$.

Definition 13. *A ring is a set R together with two operations, $+ : R \times R \to R$ and $\cdot : R \times R \to R$, called addition and multiplication, respectively, such that R is a group under addition and a monoid*[1] *under multiplication, and such that multiplication is bilinear.*

The following ring is an essential ingredient for h-adic Hopf algebras.

Definition 14. *The ring of complex formal power series is defined as*

$$\mathbb{C}[[h]] = \left\{ \sum_{n=0}^{\infty} c_n h^n \ : \ c_n \in \mathbb{C} \right\} \tag{3.12}$$

[1]The definition of a monoid is that same as that of a group, but without the requirement for inverse elements.

as a set, where h is an indeterminate parameter, only there for book-keeping purposes. The sum of two formal power series is given by

$$\sum_{n=0}^{\infty} c_n h^n + \sum_{n=0}^{\infty} d_n h^n = \sum_{n=0}^{\infty} (c_n + d_n) h^n \tag{3.13}$$

and the product is given by

$$\left(\sum_{n=0}^{\infty} c_n h^n \right) \left(\sum_{n=0}^{\infty} d_n h^n \right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} c_k d_{n-k} \right) h^n, \tag{3.14}$$

i.e., the operations are precisely as expected for power series.

It is an easy exercise to show that $\mathbb{C}[[h]]$ so defined is a indeed ring. We note again that the indeterminate parameter h serves no other purpose than book-keeping at this stage (although later we will find a way to give it a definite value and we will identify it with the Planck scale). Indeed, one might also view $\mathbb{C}[[h]]$ in a different way, without the h, namely as consisting of all functions $\mathbb{N} \to \mathbb{C}$. The convenience of the power series 'representation' is that the product is precisely what one would expect for power series. The ring $\mathbb{R}[[h]]$ of *real* power series is defined analogously, but we will here focus on the complex case, as is standard in the literature on h-adic Hopf algebras.
Although the concept of a vector space (and hence of an algebra) relies on an underlying field, which usually is $\mathbb{R}$ or $\mathbb{C}$ in phyics, it can be generalized to the case where, instead of a field, one may use a ring R, like $\mathbb{C}[[h]]$. This generalization is called an R-module.

Definition 15. *An R-module, where R is a ring, is defined by the same axioms as a vector space, but with the underlying field replaced by a ring R.*

Although a lot of results for vector spaces generalize to R-modules, it is not the case for all result. For example, an important difference between vector spaces and R-modules is that the former always have a (Hamel) basis, whereas the latter need not have one. For us, the prime example of an R-module is the following one.

Example 16. *Let V be a complex vector space. $V[[h]]$ is the $\mathbb{C}[[h]]$-module consisting of all power series in h with values in V,*

$$V[[h]] = \left\{ \sum_{n=0}^{\infty} v_n h^n \,:\, v_n \in V \right\}, \tag{3.15}$$

and with addition and scalar multiplication given by

$$\sum_{n=0}^{\infty} v_n h^n + \sum_{n=0}^{\infty} w_n h^n = \sum_{n=0}^{\infty} (v_n + w_n) h^n \tag{3.16}$$

$$\left(\sum_{n=0}^{\infty} c_n h^n \right) \left(\sum_{n=0}^{\infty} v_n h^n \right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} c_k v_{n-k} \right) h^n, \tag{3.17}$$

where $v_n, w_n \in V$ and $c_n \in \mathbb{C}$. Modules of this form are called topologically free modules.

Example 17. *Another important example is the $\mathbb{C}[[h]]$-module $V \otimes_{\mathbb{C}} \mathbb{C}[[h]]$, which we will also denote simply by $V \otimes \mathbb{C}[[h]]$. The addition of vectors in $V \otimes \mathbb{C}[[h]]$ is given by the usual addition in the complex tensor product and scalar multiplication happens in the second factor of the tensor product, i.e., for $f, g \in \mathbb{C}[[h]]$ and $v \in V$ we have*

$$f \cdot (v \otimes g) = v \otimes (fg), \tag{3.18}$$

where fg is the product in $\mathbb{C}[[h]]$, and this is extended linearly.

These two examples turn out to coincide in a lot of the cases. Before we prove this, let us remark that further notions like R-module homomorphisms and constructions like tensor products over R can be obtained from the vector space ones by obvious generalizations replacing the fields by rings.

Proposition 18. *If V is finite dimensional then $V \otimes_{\mathbb{C}} \mathbb{C}[[h]] \cong V[[h]]$ as $\mathbb{C}[[h]]$-modules.*

Proof. Define the map $\phi : V \otimes \mathbb{C}[[h]] \to V[[h]]$ on pure tensors as

$$\phi(v \otimes \sum c_n h^n) = \sum (c_n v) h^n. \tag{3.19}$$

This extends to a well-defined $\mathbb{C}$-linear map on the full tensor product space, by the universal property of the tensor product, and it is easy to see that the resulting map is also $\mathbb{C}[[h]]$-linear[2]. To show that the map is bijective, we fix a basis $(e_1, \ldots, e_n)$ of V. For injectivity it suffices to show, by $\mathbb{C}[[h]]$-linearity, that $\phi(x) = 0 \otimes 0$ only if $x = 0 \otimes 0$. So, noting that we can write a general $x \in V \otimes \mathbb{C}[[h]]$ as $x = \sum_i (e_i \otimes \sum_n c_{in} h^n)$, suppose that $\phi(x) = 0 \otimes 0$. (Note also that the i-sum here is necessarily a finite one.) Then

$$\phi(x) = \sum_i \phi\left(e_i \otimes \sum_n c_{in} h^n\right) = \sum_i \sum_n (c_{in} e_i)\, h^n = \sum_n \left(\sum_i (c_{in} e_i)\right) h^n = 0 \tag{3.20}$$

and hence $c_{in} e_i = 0$ for all n. Since $\{e_i\}$ is a basis of V, this implies that $c_{in} = 0$ for all i, n and that means that

$$x = \sum_i \left(e_i \otimes \sum_n c_{in} h^n\right) = \sum_i (e_i \otimes 0) = 0. \tag{3.21}$$

Hence ϕ is injective. Surjectivity is straightforward. For any element $y = \sum_n v_n h^n \in V[[h]]$, with $v_n \in V$ for all n, we can express $v_n = c_{in} e_i$ in terms of the basis, and then setting $x = \sum_i (e_i \otimes \sum_n c_{in} h^n)$ yields $\phi(x) = y$. As for vector spaces, a bijective linear map between R-modules is automatically an R-module isomorphism, so this completes the proof. □

Remark 19. *If V is infinite-dimensional the above proof shows that the map ϕ is not surjective, as for any $\sum_n v_n h^n$ in the image of ϕ the v_n will all lie in a finite dimensional subspace of V.*

We now turn to the generalized notion of an algebra, allowing for rings instead of fields. The definition is the obvious one and we will just refer to them as R-algebras. And similarly, the definition of a Hopf algebra can be generalized to allow for rings instead of field, leading to the notion of (Hopf) algebras over $\mathbb{C}[[h]]$.

[2]Strictly speaking the correct terminology here is not the term 'linear', which would be correct if the underlying ring was actually a field, but instead it is more common to say that ϕ is a $\mathbb{C}[[h]]$-module map. We will however also continue to refer to this property simply as $\mathbb{C}[[h]]$-linearity.

Definition 20. *An h-adic (Hopf) algebra is a (Hopf) algebra over* $\mathbb{C}[[h]]$*. An h-adic Hopf algebra which is topologically free as a* $C[[h]]$*-module, as in example 16, is called a topologically free Hopf algebra.*

Example 21. *If A is a complex algebra then* $A[[h]]$*, with the* $\mathbb{C}[[h]]$*-module structure as in example 16, has a natural h-adic algebra structure, where the multiplication is given by the obvious analog of the scalar multiplication law in* $A[[h]]$*.*

3.2.2 The h-adic Topology

Any $\mathbb{C}[[h]]$-module can be endowed with a topology called the h-adic topology. Although this is quite a technical subject, we will develop several important properties of the topology here, because it is essential for understanding some of the operations the can be used on h-adic (Hopf) algebras, like term-by-term calculations and (nonlinear) basis transformations.

The first construction that we need to discuss is the h-adic completion of a $\mathbb{C}[[h]]$-module. Let M be a $\mathbb{C}[[h]]$-module. Consider the ideals $h^n M$ of M for each $n = 0, 1, 2, \dots$ and note that we have canonical projections

$$\pi_{n+1} : M/h^{n+1}M \to M/h^n M = \left(M/h^{n+1}M\right)/h^n M \tag{3.22}$$

for all $n \geq 0$. This is a case where the notion of *inverse limit* is applicable. This notion extends to much more general situations, but here we only care about applying it to our situation. We define a *coherent sequence* as an element of the Cartesian product $x \in \prod_{n=0}^{\infty}(M/h^n M)$, written $x = (x_0, x_1, \dots)$ which satisfies $\pi_{n+1}(x_{n+1}) = x_n$ for each $n \geq 1$. The subset of the Cartesian product consisting of coherent sequences is called the *inverse limit* and is denoted by

$$\widetilde{M} = \varprojlim (M/h^n M)\,. \tag{3.23}$$

The inverse limit $\varprojlim (M/h^n M)$ has a natural $\mathbb{C}[[h]]$-module structure: both addition and scalar multiplication are evaluated pointwise, i.e., if $f \in \mathbb{C}[[h]]$ and $(x_i), (y_i) \in \varprojlim (M/h^n M)$ we define

$$f \cdot (x_0, x_1, \dots) + (y_0, y_1, \dots) = (fx_0 + y_0, fx_1 + y_1, \dots) \tag{3.24}$$

where it must be noted that in the i^{th} component, f is interpreted as $f + h^{i+1}\mathbb{C}[[h]] \in \mathbb{C}[[h]]/h^{i+1}C[[h]]$. We call $\varprojlim (M/h^nM)$ the (h-adic) *completion* of M. Note that there is a natural $\mathbb{C}[[h]]$-module morphism $i : M \to \varprojlim (M/h^nM)$ given by $M \ni x \mapsto (x + M, x + hM, x + h^2M, \dots)$. We say that M is an (h-adically) complete $\mathbb{C}[[h]]$-module if this map is an isomorphism[3]. It is an intstructive exercise to check that $V[[h]]$ as in example 16 is complete.

Any $\mathbb{C}[[h]]$-module M can be endowed with the h-adic topology, which is defined by means of the neighborhood base $\mathscr{U}_x = \{x + h^nM \ : \ n \in \mathbb{N}\}$ at each point $x \in M$. When M is complete, it follows by injectivity of i that $\bigcap_n h^nM = 0$ which implies that the h-adic topology is Hausdorff. In this case it coincides with the metric topology coming from the metric $d(x, y) = 2^{-k}$, where k is the smallest integer such that $x - y \notin h^kM$, with $k = \infty$ if no such integer exists, which then is indeed a metric on M in the sense that it has values in $[0, \infty)$, it is symmetric, satisfies the triangle inequality, and satisfies $d(x, y) = 0 \Leftrightarrow x = y$. An essential property of the h-adic topology is given by the following proposition and its corollary.

Proposition 22. *Any $\mathbb{C}[[h]]$-module homomorphism $\phi : M \to N$ between $\mathbb{C}[[h]]$-modules M, N is automatically continuous in the h-adic topology, which implies that $f(m_n) \to f(m)$ in N whenever $m_n \to m$ in M.*

Proof. Open neighborhood bases at $x \in M$ and $\phi(x) \in N$ for the topologies on M and N, respectively, are given by $\mathscr{U} = \{x + h^nM \ : \ n \in \mathbb{N}\}$ and $\mathscr{V} = \{\phi(x)+h^nN \ : \ n \in \mathbb{N}\}$. Now for any open neighborhood $\phi(x)+h^nN \in \mathscr{V}$ we pick the open neighborhood $x+h^nM \in \mathscr{U}$, and then $\mathbb{C}[[h]]$-linearity, together with the fact that $\phi(M) \subset N$, implies that $\phi(x + h^nM) = \phi(x) + h^n\phi(M) \subset \phi(x) + h^nN$. This shows that ϕ is continuous at x, which was an arbitrary point. □

Corollary 23. *A $\mathbb{C}[[h]]$-linear map $\phi : V[[h]] \to W[[h]]$, where V, W are complex vector spaces may be evaluated term by term, i.e., if $f = \sum_{i=0}^{\infty} v_ih^i \in V[[h]]$ then $\phi(f) = \sum_{i=0}^{\infty} \phi(v_i)h^i$.*

[3]The terminology differs per author. For example, in [30] the term *complete* is defined to mean only that the map i is surjective, not necessarily bijective, whereas if the complementary property, that i is injective, holds, then M is called *separated*. Hence a lot of results in [30] are about complete, separated modules.

Proof. For any partial sum $f_n = \sum_{i=0}^{n} v_i h^i$ we have

$$\phi(f_n) = \sum_{i=0}^{n} \phi(v_i h^i) = \sum_{i=0}^{n} \phi(v_i) h^i, \tag{3.25}$$

by $\mathbb{C}[[h]]$-linearity. This shows that $\phi(f_n)$ converges in the h-adic topology to $\sum_{i=0}^{\infty} \phi(v_i) h^i$. At the same time, since ϕ is $\mathbb{C}[[h]]$-linear, it is automatically continuous in the h-adic topology, by prop. 22, and hence, since f_n converges in the h-adic topology to f, $\phi(f_n)$ converges to $\phi(f)$. But $W[[h]]$ is complete and hence Hausdorff. Since sequences in Hausdorff spaces can only have one limit, so we must have $\phi(f) = \sum_{i=0}^{\infty} \phi(v_i) h^i$. □

3.2.3 Finitely Generated h-adic (Hopf) Algebras

Just like a set of generators with (polynomial) relations can be used to define an algebra, this kind of data can also be used to define an h-adic algebra, and the interesting new aspect in the h-adic case is that the relations need not be polynomial but can contain power series in them. Here we introduce the notion of *the h-adic algebra generated by the set* $S = \{1, T_1, \dots T_n\}$ *of generators with relations* $\phi_i = 0$. To this end one first constructs the tensor algebra $T(\mathrm{span}_{\mathbb{C}} S)$ of the complex vector space spanned by n basis vectors which we denote as T_i, $i = 1, \dots n$. Next one constructs the algebra of power series $T(\mathrm{span}_{\mathbb{C}} S)[[h]]$ in h with coefficients in the tensor algebra. And finally the desired relations in this tensor algebra are then enfored by taking the quotient of this space with respect to *the h-adic closure* of the ideal $\mathbb{C}[[h]]\langle\phi_i\rangle$ generated (over $\mathbb{C}[[h]]$) by the relations ϕ. The result is the desired h-adic algebra A:

$$A = T(\mathrm{span}_{\mathbb{C}} S)[[h]] / \widetilde{\mathbb{C}[[h]]\langle\phi_i\rangle}. \tag{3.26}$$

By construction, all the relations $\phi_i = 0$ will indeed be satisfied in A. It should now also be clear that the relations ϕ_i may contain any element in $T(\mathrm{span}_{\mathbb{C}} S)[[h]]$.

Given a $\mathbb{C}[[h]]$-module homomorphism $\phi : A \to B$ between topologically free algebras A, B, we have already seen that it is allowed to evaluate ϕ term by term on power series. In the case that A is finitely generated, and ϕ is also an

algebra homomorphism, we are allowed to do even more. Consider an element $a \in A$. It can be written in terms of the generators g_i as a function of the form $a = f(\{g_j\}) = \sum_{i=0}^{\infty} P_i(\{g_j\})h^i$, where each P_i is a polynomial in the generators. Now by the preceding comment, we may evaluate ϕ on a by evaluating it term by term, i.e., on each polynomial P_i separately. But because ϕ is an algebra map and P_i a polynomial, we can bring ϕ inside the P_i and act on each generator individually, $\phi\left(P_i(\{g_j\})\right) = P_i(\{\phi(g_j)\})$. Combining these two facts, we conclude that we have

$$\phi\left(f(\{g_j\})\right) = f\left(\{\phi(g_j)\}\right). \tag{3.27}$$

To put it briefly,

Proposition 24. *The homomorphism ϕ may be evaluated on each generator individually.*

This shows that an algebra morphism is completely specified by its actions on generators. The same holds for algebra anti-morphisms, like the antipode of an (h-adic) Hopf algebra. In particular, once a finitely generated h-adic algebra is specified, to define an h-adic Hopf algebra structure on it one only needs to define the coproduct, co-unit and antipode on the generators of the algebra. The following instructive example illustrates these ideas.

Proposition 25. *If a is a primitive element in a topologically free Hopf algebra $V[[h]]$, meaning that $\Delta a = a\otimes 1 + 1\otimes a$, then e^{ah} is a group like element, meaning that $\Delta(e^{ah}) = e^{ah} \otimes e^{ah}$.*

Proof. First not that e^{ah} is a well-defined element in $V[[h]]$ since it can be expressed as a power series in h with the coefficients being polynomials in the generators. Using cor. 23, we have $\Delta\left(e^{ah}\right) = e^{\Delta(a)h} = e^{ah\otimes 1 + 1\otimes ah}$. Now note that $ah\otimes 1$ and $1\otimes ah$ commute, so that the exponential factorizes, $\Delta\left(e^{ah}\right) = e^{ah\otimes 1}e^{1\otimes ah}$, and then, writing out the power series it is easy to see that $e^{ah\otimes 1} = e^{ah}\otimes 1$ and $e^{1\otimes ah} = 1\otimes e^{ah}$, so that we obtain the final result $\Delta\left(e^{ah}\right) = \left(e^{ah}\otimes 1\right)\left(1\otimes e^{ah}\right) = e^{ah}\otimes e^{ah}$. □

3.3 Nonlinear Basis Transformations

3.3.1 Lie Algebra Basis Transformations

In the (physics) literature on Hopf algebras one is often interested in a certain Hopf algebra in a specific 'basis'. Basis is in quotes, because we are not talking about a vector space basis, or even a $\mathbb{C}[[h]]$-module basis, but when one talks about a basis of an h-adic Hopf algebra, one usually means the set of generators of the algebra and there relations.

We will compare the situation with the theory of Lie algebras, because in a certain sense our current situation is a generalization of this. Any given Lie algebra can be expressed in terms of a different vector space basis, and in each basis the Lie brackets will in general be different. The Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ for instance can be defined as the complex vector space with basis J_1, J_2, J_3 and Lie brackets given by linear extension of $[J_i, J_j] = \epsilon_{ijk}J_k$, but it can just as well be defined as the complex vector space with basis K_+, K_-, K_3 and Lie brackets given by linear extension of $[K_+, K_-] = -2K_3, [K_\pm, K_3] = \pm K_\mp$. That these two descriptions actually define the same Lie algebra is seen simply by noting that the two are related by a basis transformation

$$K_\pm = J_1 \pm J_2, \quad K_3 = J_3. \tag{3.28}$$

To make this a bit more formal (which will allow us to generalize these ideas to the realm of h-adic Hopf algebras), we can say that the two versions define the same Lie algebra because there exists a Lie algebra isomorphism between the two Lie algebras, given by

$$\phi(J_1) = \frac{1}{2}\left(K_+ + K_-\right), \qquad \phi(J_2) = \frac{1}{2}\left(K_+ - K_-\right), \qquad \phi(J_3) = K_3. \tag{3.29}$$

That this is indeed a Lie algebra isomorphism is readily checked; the map is invertible and satisfies

$$[\phi(J_i), \phi(J_j)] = \phi\left([J_i, J_j]\right) \tag{3.30}$$

for $i = 1, 2, 3$. In the context of Lie algebras a basis change is, of course, always a linear transformation. In the context of h-adic (Hopf) algebras, we are

not restricted to linear transformations anymore. Let us first see an example. Consider the universal enveloping $U(\mathfrak{sl}(2,\mathbb{C}))$ and extend it to an h-adic algebra by looking at $U(\mathfrak{sl}(2,\mathbb{C}))[[h]]$, the algebra of formal power series with values in the enveloping algebra. This object has a natural h-adic algebra structure, as introduced in example 16. We can again describe this algebra in multiple ways. We can say that it is generated by the J_i or by the $K_\pm, K_3$. The two descriptions are of course equivalent, and indeed, the isomorphism ϕ from earlier extends to an isomorphism between the two h-adic algebras. To see this we first define ϕ, as before, on the complex span of the generators J_i (and we interpret the image of ϕ as lying in the alternative version of $U(\mathfrak{sl}(2,\mathbb{C}))[[h]]$). Then, being a linear map on this span, ϕ automatically extends uniquely to a complex algebra homomorphism on the tensor algebra, and then it automatically extends uniquely a $\mathbb{C}[[h]]$ algebra homomorphism (this is easy to verify) on the space of power series with values in the tensor algebra. Now we have to check that ϕ respects the relations that we impose on those power series when generating the $U(\mathfrak{sl}(2,\mathbb{C}))[[h]]$, because then ϕ will remain well-defined after quotienting out the relations. But the statement that ϕ respects these relations is precisely the statement, which we checked above, that ϕ is a Lie algebra homomorphism. Hence ϕ extends to a well-defined $\mathbb{C}[[h]]$-algebra homomorphism between the two versions of $U(\mathfrak{sl}(2,\mathbb{C}))[[h]]$. To construct the inverse of ϕ, we simply take the inverse of the Lie algebra version of ϕ, and then we extend it in the same way to a $\mathbb{C}[[h]]$-algebra homomorphism. Let us call the resultant map ψ and see that those two maps are indeed inverses of each other. By definition, for the generators J_i, we have $\psi(\phi(J_i)) = J_i$. By the homomorphism property, $\psi \circ \phi$ also acts as the identity on sums of products of generators. And because, as proven in prop. 22, $\psi \circ \phi$ is continuous, it also acts as the identity on infinite power series. Hence $\psi \circ \phi = \mathrm{id}$ and the maps are indeed inverses of each other. We have proven that the two versions of the h-adic enveloping algebras of $\mathfrak{sl}(2,\mathbb{C})$ are isomorphic, as expected. Clearly we used no special properties of $\mathfrak{sl}(2,\mathbb{C})$ in the proof, and no special facts about the specific isomorphism ϕ. It is easily seen that the proof extends to

Proposition 26. *If $\phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra isomorphism then ϕ extends uniquely to a $\mathbb{C}[[h]$-algebra isomorphism $U(\mathfrak{g})[[h]] \to U(\mathfrak{h})[[h]]$.*

3.3.2 Flash Forward: The κ-Poincaré Hopf Algebra

At this point it is clear that the theory of Lie algebras is embedded in the theory of h-adic (Hopf) algebras. We are now ready to see one of the new possibilities that opens up when we work within the realm of h-adic algebras. We will for the first time see an example of a so-called nonlinear basis transformation. We will treat the example of the κ-Poincaré Hopf algebra in $1+1$ dimensions, denoted[4] $H = U_h(\mathfrak{iso}(1,1))$, which, as an h-adic algebra, is generated by P_0, P_1, N with relations

$$[P_0, P_1] = 0, \qquad [N, P_0] = P_1, \qquad [N, P_1] = \kappa \sinh(P_0/\kappa), \tag{3.31}$$

where $\kappa = 1/h$. We choose to work with κ here because it is the most common in the physics literature. We will later identify it with the Planck scale. The third commutation relation, which would not be allowed in the realm of Lie algebras, is allowed here because

$$\kappa \sinh\left(\frac{P_0}{\kappa}\right) = P_0 + \frac{P_0^3}{6\kappa^2} + \frac{P_0^5}{120\kappa^4} + \dots \tag{3.32}$$

is a well-defined power series in $1/\kappa$ with coefficients in the tensor algebra of the span of the generators. Note also that when we take $\kappa \to \infty$, which in the context of power series in $1/\kappa$ means simply that we evaluate all power series only to zeroth order in $1/\kappa$, the relations reduce to those of the Poincaré algebra. We will defer the treatment of the coalgebra sector to section 3.5, where we officially introduce the κ-Poincaré Hopf algebra.

A nonlinear basis transformation can now be defined in terms of a new set of generators, that can be chosen (*a priori)* freely from the algebra. In this example we choose the following set of alternative generators,

$$\tilde{P}_0 = P_0, \qquad \tilde{P}_1 = P_1 e^{-P_0/2\kappa}, \qquad \tilde{N} = N e^{-P_0/2\kappa}, \tag{3.33}$$

The first important observation is that this correspondence is 'invertible' in the sense that each of the original P_0, P_1, N can be consistently expressed as a power

[4] ISO(1,1) is a common notation for the Poincaré group in $1+1$ dimensions, the isometry group of $1+1$ dimensional Minkowski space. $\mathfrak{iso}(1,1)$ is its Lie algebra, the Poincaré algebra, and U_h stands for a deformation of this algebra with deformation parameter h.

series in $1/\kappa$ with values in the tensor algebra of the transformed generators $\tilde{P}_0, \tilde{P}_1, \tilde{N}$, namely as

$$P_0 = \tilde{P}_0, \qquad P_1 = \tilde{P}_1 e^{\tilde{P}_0/2\kappa}, \qquad N = \tilde{N} e^{\tilde{P}_0/2\kappa}. \tag{3.34}$$

This is the property that allows us to speak of a 'basis transformation'. We now want to relate the original algebra H – the one generated by P_0, P_1, N with the stated relations – to the algebra generated by $\tilde{P}_0, \tilde{P}_1, \tilde{N}$ with suitable alternative relations, let us call it H'. In this language, we must formulate the above redefinition of the generators a bit differently to make sense of it. We start by defining a map $\phi : H \to H'$ corresponding to the relation between the generators above,

$$\phi : P_0 \mapsto \tilde{P}_0, \qquad P_1 \mapsto \tilde{P}_1 e^{\tilde{P}_0/2\kappa}, \qquad N \mapsto \tilde{N} e^{\tilde{P}_0/2\kappa}. \tag{3.35}$$

We speak of a *nonlinear* basis transformation, because ϕ definitely appears to be nonlinear in the generators. And although this is a correct observation, we should realize that actually the map ϕ is a perfectly linear map when viewed in the context of $\mathbb{C}[[h]]$-modules! The element to which P_1 is mapped, $P_1 e^{-P_0/2\kappa}$, is just some element in H', which is a $\mathbb{C}[[h]]$-module and in particular a complex vector space. The same holds for the elements to which the other generators are mapped. In fact it is even the case that ϕ extends to a well-defined linear map $H \to H'$ which turns out to be an isomorphism when we adopt for H' the relations below, so that the two algebras, although formulated in terms of different sets of generators, are the in fact identical. This follows from the general result in prop. 27 which we will prove soon. So a great advantage of working over the ring of complex power series is that things which would usually be considered nonlinear now become in fact linear and therefore relatively easy to deal with. Now let us demonstrate how to find the form of the algebra in the 'new basis', in practice. The result will be called the *bicrossproduct basis* of the κ-Poincaré algebra.

What we want to know are the commutation relations between the new generators $\tilde{P}_0, \tilde{P}_1, \tilde{N}$. Two of those are trivially found in the following way

$$[\tilde{P}_0, \tilde{P}_1] = [P_0, P_1 e^{-P_0/2\kappa}] = [P_0, P_1]e^{-P_0/2\kappa} + P_1[P_0, e^{-P_0/2\kappa}] = 0 + 0 = 0, \quad (3.36)$$

$$[\tilde{N}, \tilde{P}_0] = [N e^{-P_0/2\kappa}, P_0] = [N, P_0]e^{-P_0/2\kappa} + 0 = P_1 e^{-P_0/2\kappa} = \tilde{P}_1. \quad (3.37)$$

The third one requires some more work, but is very instructive.

$$[\tilde{N}, \tilde{P}_1] = [N e^{-P_0/2\kappa}, P_1 e^{-P_0/2\kappa}] \quad (3.38)$$

$$= N[e^{-P_0/2\kappa}, P_1]e^{-P_0/2\kappa} + P_1[N, e^{-P_0/2\kappa}]e^{-P_0/2\kappa} + [N, P_1]e^{-P_0/\kappa} \quad (3.39)$$

$$= P_1[N, e^{-P_0/2\kappa}]e^{-P_0/2\kappa} + [N, P_1]e^{-P_0/\kappa} \quad (3.40)$$

where the first of the three commutators vanishes because P_0 commutes with P_1. The two remaining commutators will be examined individually. The second one is simple,

$$[N, P_1]e^{-P_0/\kappa} = \kappa \sinh(P_0/\kappa)e^{-P_0/\kappa} = \frac{\kappa}{2}\left(1 - e^{-2P_0/\kappa}\right) = \frac{\kappa}{2}\left(1 - e^{-2\tilde{P}_0/\kappa}\right), \quad (3.41)$$

and for the first one we have to work term by term in the power series. Since the commutator is just a certain linear combination of algebra products, and because the algebra product is $\mathbb{C}[[h]]$-bilinear, and hence h-adically continuous in each slot, the commutator is also h-adically continuous in each slot, which justifies

the term-by-term computation. Hence we get

$$[N, e^{-P_0/2\kappa}] = \left[N, \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{(-2\kappa)^n} P_0^n\right] = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{(-2\kappa)^n} [N, P_0^n] \tag{3.42}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!} \frac{1}{(-2\kappa)^n} \left(P_0^{n-1}[N, P_0] + P_0^{n-2}[N, P_0]P_0 + \cdots + [N, P_0]P_0^{n-1}\right) \tag{3.43}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!} \frac{1}{(-2\kappa)^n} \left(P_0^{n-1}P_1 + P_0^{n-2}P_1P_0 + \cdots + P_1P_0^{n-1}\right) \tag{3.44}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!} \frac{1}{(-2\kappa)^n} n P_0^{n-1}P_1 \qquad \text{(since } P_0 \text{ and } P_1 \text{ commute)} \tag{3.45}$$

$$= \frac{-1}{2\kappa} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \frac{1}{(-2\kappa)^{n-1}} P_1 P_0^{n-1} = \frac{-1}{2\kappa} \left(\sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{(-2\kappa)^n} P_0^n\right) P_1 \tag{3.46}$$

$$= \frac{-1}{2\kappa} \left(\sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{(-2\kappa)^n} P_0^n\right) P_1 = \frac{-1}{2\kappa} e^{-P_0/2\kappa} P_1, \tag{3.47}$$

which leads to the term

$$P_1[N, e^{-P_0/2\kappa}]e^{-P_0/2\kappa} = \frac{-1}{2\kappa} P_1 e^{-P_0/2\kappa} P_1 e^{-P_0/2\kappa} = \frac{-1}{2\kappa} \tilde{P}_1^2. \tag{3.48}$$

Hence the relations in the new basis are found to be

$$[\tilde{P}_0, \tilde{P}_1] = 0, \qquad [\tilde{N}, \tilde{P}_0] = \tilde{P}_1, \qquad [\tilde{N}, \tilde{P}_1] = \frac{\kappa}{2}\left(1 - e^{-2\tilde{P}_0/\kappa}\right) - \frac{1}{2\kappa}\tilde{P}_1^2. \tag{3.49}$$

The algebra h-adically generated by $\tilde{P}_0, \tilde{P}_1$ and $\tilde{N}$ with these relations is isomorphic to the κ-Poincaré algebra we started with, and the two equivalent algebras are related by what we call a (nonlinear) basis transformation. The basis arrived at here is called the bicrossproduct basis and is the one that is most frequently considered in physics applications. It was derived for the first time in [13]. It is interesting to note that this is also precisely the 'algebra' that is satisfied by

the spacetime translation generators (i.e. KVFs) $(\tilde{P}_0, \tilde{P}_1)$ and the momentum space boost $(\tilde{N})$ in the case that spacetime is described with the Minkowski metric and momentum space has a de Sitter metric, which the interested reader can check. This immediately suggests that if the κ-Poincaré algebra is to describe symmetries of some physics model, then it would have something to do with a de Sitter momentum space manifold. This is indeed the current consensus.

We now prove the general result on basis transformations of h-adic algebras.

Proposition 27. *(Nonlinear Basis Transformations)*
Let A be the h-adic algebra generated by $\{g_1, g_2, \ldots, g_n\}$ with commutator relations $[g_i, g_j] = G_{ij}$ for some G_{ij}, and let a map $\phi : \{g_1, g_2, \ldots, g_n\} \to \tilde{A}_{pre}$ be given into $\tilde{A}_{pre}$, the algebra h-adically generated by $\{\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_n\}$ with no relations (yet). Assume that ϕ, when extended linearly to all of A, is invertible on generators, in the sense that there exists a $\mathbb{C}[[h]]$-linear map $\phi^{-1} : \tilde{A}_{pre} \to A$ such that $\phi^{-1} \circ \phi(g_i) = g_i$ and $\phi \circ \phi^{-1}(\tilde{g}_i) = \tilde{g}_i$ for all i. Define commutator relations on $\tilde{A}_{pre}$ schematically as

$$[\tilde{g}_i, \tilde{g}_j] = comm_\sim(\tilde{g}_i, \tilde{g}_j), \tag{3.50}$$

$$comm_\sim \equiv \phi \circ comm \circ (\phi^{-1} \times \phi^{-1}), \tag{3.51}$$

where comm denotes the commutator in the original algebra A. In other words, we simply use ϕ^{-1} to express the alternative generators in terms of the original g_i, then we compute the commutator using the relations among the g_i of the original algebra, and then we use ϕ to translate back to the $\tilde{g}_i$. Then the algebra $\tilde{A}$ generated by $\{\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_n\}$ with these commutator relations is isomorphic to A. We say that A and $\tilde{A}$ are related by a (possibly nonlinear) basis transformation. (For technical reasons we also assume that the commutator of any two elements in $\tilde{A}_{pre}$, not only generators, is given by the expression $comm_\sim$.)

Proof. Extend the map ϕ, given on the basis $\{g_1, g_2, \ldots, g_n\}$, to a $\mathbb{C}$-linear map on the span V of the generators. Then ϕ can be further extended first to a complex algebra map on the tensor algebra $T(V)$ of V, and then even to a $\mathbb{C}[[h]]$ algebra map on $T(V)[[h]]$. To show that this induces an algebra homomorphism $A \to \tilde{A}_{\text{pre}}$ we then only need to show that ϕ is compatible with the relations on A, i.e., $[\phi(g_i), \phi(g_j)] = \phi(G_{ij})$, because A is the quotient of $T(V)[[h]]$ by the

(closure of the) algebra generated by the relations. Schematically this comes down to the condition that

$$\mathrm{comm}_\sim \circ (\phi \times \phi) = \phi \circ \mathrm{comm}, \tag{3.52}$$

and it follows simply from the definition of $\mathrm{comm}_\sim$ above that this condition is satisfied. Hence ϕ is an algebra morphism $A \to \tilde{A}_{\mathrm{pre}}$, which of course induces an algebra morphism $\phi : A \to \tilde{A}$.
We can do this whole construction in the opposite direction as well, which leads to an algebra morphism $\phi' : \tilde{A} \to A$. It can then be seen that ϕ and ϕ' are mutually inverse maps (on generators and hence everywhere), proving that $\tilde{A} \cong A$. □

As a remark, note that the inverse ϕ^{-1} is usually easily found, as can be seen for instance in the κ-Poincaré example above.

3.4 Phase Space Representations and Physics

Phase space representations provide a way to connect the mathematical structure of a given h-adic Hopf algebra to actual physics. Since finitely generated h-adic Hopf algebras like the κ-Poincaré algebra can be thought of as deformations of Lie algebras, a representation of a Hopf algebra should be defined in a similar way as that of a Lie algebra. But it is not enough to define a phase space representation of a finitely generated h-adic Hopf algebra as a linear map to the smooth functions on phase space such that the relations among the generators are satisfied (which would be the closest analog of a Lie algebra representation), because then we are neglecting the additional structure provided by Δ, ϵ and S. In particular, we would like to retain in the representation the fact that these maps may be evaluated term by term in power series expressions, e.g. $\Delta(e^{ah}) = e^{\Delta(a)h}$. To ensure this we will need a suitable definition of a phase space representation and we will provide this definition in this section.

3.4.1 Phase Space Representations

Why not an ordinary algebra representation?

It is important to realize that a 'standard' definition of representation of an algebra (in contrast to *Lie* algebra) will not work when we want to work on

phase space. An essential requirement, coming from the Lie algebra analog, is that the commutation relations between the generators of the algebra need to be satisfied on phase space not as commutators with respect to Poisson brackets. For instance, the relation $[E, H] \equiv EH - HE = H$ should be represented on phase space as $\{E, H\} = H$. If we would also require that the representation, say ϕ, was a true algebra representation, and hence an algebra homomorphism, we would have

$$\phi(H) = \phi([E, H]) = \phi(EH - HE) = \phi(E)\phi(H) - \phi(H)\phi(E) \tag{3.53}$$
$$= \phi(E)\phi(H) - \phi(E)\phi(H) = 0 \tag{3.54}$$

because $\phi(E)$ and $\phi(H)$ are *functions* and hence they necessarily commute. Thus operators like H, which are equal to a commutator, would automatically vanish in the representation. In this way we would lose a lot of information. Hence requiring ϕ to be an algebra homomorphism is not a good idea. We conclude that a h-adic algebra representation on phase space must be something different than a usual algebra representation. But of course we do want to retain as much structure of the original Hopf algebra as possible in representation.
A way to achieve this is, in most cases, is to first *forget about the relations among the generators* and represent the resulting tensor algebra (as opposed to its quotient with the relations) as we would usually represent an algebra. This then keeps track of the algebra structure. The relations among the generators will be incorporated in a different way, namely in the Poisson brackets, so it is acceptable to forget about them when dealing with algebra multiplication. When we represent the algebra sector of a Hopf algebra in this way, it turns out that (modulo some mild technicalities) there is a natural way to define also the coproduct, co-unit and antipode on phase space. We will now go through the technical definitions and proofs, and then apply it to the κ-Poincaré Hopf algebra.
Before we define phase space representation we first need to specify the class of Hopf algebras to which our definition will be applicable.

Definition 28. *A finitely generated h-adic Hopf algebra H is called representable if the maps Δ, ϵ and S can be extended to the whole tensor algebra (instead of just the quotient of the tensor algebra, where the maps are originally defined), i.e., $(T(V)[[h]], \tilde{\Delta}, \tilde{\epsilon}, \tilde{S})$ is a Hopf algebra such that H is a Hopf subalgebra thereof.*

Moreover we assume that the coproduct takes values in $T(V)[[h]] \otimes T(V)[[h]] \subset (T(V) \otimes T(V))[[h]]$. *We call this extended Hopf algebra the extension of* H *and denote it by* $\tilde{H}$.

For the Hopf algebras like the κ-Poincaré Hopf algebra and many other Hopf algebra deformations of classical Lie algebras these conditions are met.

Definition 29. *A phase space representation of a representable Hopf algebra* H *is a map* $T : H \to \mathscr{F}(M)$, *where* M *is a phase space (i.e., a smooth Poisson manifold) and* $\mathscr{F}(M)$ *the space of smooth functions* $M \to \mathbb{C}$, *together with a complex number* $h_0 \in \mathbb{C}$, *such that the relations between the Hopf algebra generators are realized on phase space through the Poisson bracket and such that if* $a \in H$ *can be expressed as*

$$a = \sum_i \left(\sum_j c_j \prod_k g_k \right) h^i, \tag{3.55}$$

where g_k *are generators of* H, *then*

$$T(a) = \sum_i \left(\sum_j c_j \prod_k T(g_k) \right) h_0^i, \tag{3.56}$$

i.e. any formal power series in the indeterminate h *is mapped to the corresponding actual power series in* h_0 *in* $\mathscr{F}(M)$. *This means in particular that* T *is a complex algebra map.*

Heuristically this says that T is even $\mathbb{C}[[h]]$-linear and continuous in some sort of h-adic topology. The definition of course requires that all actual power series in (3.56) must be convergent everywhere in phase space. Note that the mapping of the *indeterminate* h to an actual complex number h_0 is needed in order to be able to do actual calculations in physics. This covers the algebra sector. Next we need to assign a coproduct to our phase space representations. Let T be a representation of H. Define the coproduct via

$$\Delta(T(a)) = (T \otimes T)(\Delta(a)) = T(a^{(1)}) \otimes T(a^{(2)}), \tag{3.57}$$

where $T \otimes T$ is the map $H \otimes H \to \mathscr{F}(M) \otimes \mathscr{F}(M)$ induced by T via

$$(T \otimes T)(a \otimes b) = T(a) \otimes T(b). \tag{3.58}$$

We now show that this coproduct indeed still satisfies its usual coassociativity property. We have

$$\begin{aligned}
(1 \otimes \Delta) \circ \Delta(T(a)) &= (1 \otimes \Delta)(T(a^{(1)}) \otimes T(a^{(2)})) && (3.59)\\
&= T(a^{(1)}) \otimes T(a^{(2)(1)}) \otimes T(a^{(2)(2)}) && (3.60)\\
&= (T \otimes T \otimes T)(a^{(1)} \otimes a^{(2)(1)} \otimes a^{(2)(2)}) && (3.61)\\
&= (T \otimes T \otimes T)((1 \otimes \Delta) \circ \Delta(a)) && (3.62)\\
&= (T \otimes T \otimes T)((\Delta \otimes 1) \circ \Delta(a)) && (3.63)\\
&= \ldots = (\Delta \otimes 1) \circ \Delta(T(a)). && (3.64)
\end{aligned}$$

Hence

$$(1 \otimes \Delta) \circ \Delta = (\Delta \otimes 1) \circ \Delta \tag{3.65}$$

in the representation. We define the co-unit $\epsilon : T(H) \to \mathbb{C}$ as $\epsilon(T(a)) = \epsilon(a)$. Then we have

$$\begin{aligned}
(\epsilon \otimes \mathrm{id}) \circ \Delta(T(a)) &= (\epsilon \otimes \mathrm{id})(T(a^{(1)}) \otimes T(a^{(2)})) = \epsilon(a^{(1)})T(a^{(2)}) && (3.66)\\
&= T(\epsilon(a^{(1)})a^{(2)}) = T((\epsilon \otimes \mathrm{id}) \circ \Delta(a)) = T(a) && (3.67)
\end{aligned}$$

which proves that our phase space representation is a coalgebra. Next we make

sure that Δ, ϵ and S are algebra (anti)homomorphisms.

$$\begin{aligned}
\Delta(T(a)T(b)) &= \Delta(T(ab)) = (T\otimes T)(\Delta(ab)) = (T\otimes T)(\Delta(a)\Delta(b)) && (3.68)\\
&= (T\otimes T)\left(\left(a^{(1)}\otimes a^{(2)}\right)\left(b^{(1)}\otimes b^{(2)}\right)\right) && (3.69)\\
&= (T\otimes T)\left(a^{(1)}b^{(1)}\otimes a^{(2)}b^{(2)}\right) && (3.70)\\
&= T\left(a^{(1)}b^{(1)}\right)\otimes T\left(a^{(2)}b^{(2)}\right) && (3.71)\\
&= T\left(a^{(1)}\right)T\left(b^{(1)}\right)\otimes T\left(a^{(2)}\right)T\left(b^{(2)}\right) && (3.72)\\
&= \left(T\left(a^{(1)}\right)\otimes T\left(a^{(2)}\right)\right)\left(T\left(b^{(1)}\right)\otimes T\left(b^{(2)}\right)\right) && (3.73)\\
&= (T\otimes T)(\Delta a)(T\otimes T)(\Delta b) = \Delta(T(a)\Delta(T(b)), && (3.74)
\end{aligned}$$

$$\epsilon(T(a)T(b)) = \epsilon(T(ab)) = \epsilon(ab) = \epsilon(a)\epsilon(b) = \epsilon(T(a))\epsilon(T(b)), \tag{3.75}$$

$$\begin{aligned}
S(T(a)T(b)) &= S(T(ab)) = T(S(ab)) = T(S(b)S(a)) && (3.76)\\
&= T(S(b))T(S(a)) = S(T(b))S(T(a)). && (3.77)
\end{aligned}$$

Hence the representation is a bialgebra. Finally the antipode S, which we define as

$$S(T(a)) = T(S(a)), \tag{3.78}$$

satisfies

$$\begin{aligned}
\mu\circ(\mathrm{id}\otimes S)\circ\Delta(T(a)) &= \mu\circ(\mathrm{id}\otimes S)(T(a^{(1)})\otimes T(a^{(2)})) && (3.79)\\
&= T(a^{(1)})T(S(a^{(2)})) = T(a^{(1)}S(a^{(2)})) && (3.80)\\
&= T(\mu\circ(1\otimes S)\circ\Delta(a)) = T(\eta\circ\epsilon(a)) && (3.81)\\
&= T(\epsilon(a)1) = \epsilon(a)T(1) = \epsilon(a)1 = \epsilon(T(a))1 && (3.82)\\
&= \eta\circ\epsilon(T(a)), && (3.83)
\end{aligned}$$

which completes the list of Hopf algebra relations. Technically there is one more thing we should take care of. We must define the actual function space that we will use. Note that not every formal power series will be mapped to well-defined function $M\to\mathbb{C}$. For that reason, we should restrict to some function space that is closed under all maps like product, coproduct, etc. We will however neglect

this technical issue, as it will not play a role in our applications. Then, up to this technicality, we have proven the following.

Proposition 30. *A phase space representation of a Hopf algebra is itself a commutative Hopf algebra with the maps Δ, ϵ, S as constructed above.*

The following result, which follows from prop. 24, applies in particular when we deal with the κ-Poincaré algebra in section 3.5.

Proposition 31. *Let S be the antipode of a representation T of a Hopf algebra H on phase space T^*M, with M some manifold, and suppose that in a certain chart the coordinate functions p_μ of M satisfy $p_\mu = T(P_\mu)$, where P_μ are (some of the) generators of H. Let $f = T(a) \in T(H)$ and assume that the power series expansion of $a \in H$ contains only the generators P_μ and no other ones (Note that we are allowed to say 'the' expansion because there are no relations in the tensor algebra). Then, with P being the 'vector' consisting of the P_μ, and writing $f = T(a(P)) \equiv f(P)$ and with the obvious meaning of the function $f(S(P)) \in T(H)$, we have, for all $p \in M$,*

$$S(f(P)) = f(S(P)) \equiv f \circ (S(P)). \tag{3.84}$$

Analogous results hold for the coproduct Δ and co-unit ϵ. In words: evaluating one of the structure maps Δ, ϵ and S of the represented algebra on some given function, produces the same result as evaluating the structure map on the (abstract) generators individually and then evaluating the original function on the resulting (represented) Hopf algebra elements. In the following sections we will investigate the case in which the premises of prop. 31 are satisfied.

3.4.2 The Composition Law

All the technical work of section 3.4.1 will now begin to pay off. In the remainder of this chapter we prove that under suitable circumstances, namely those of prop. 31, a Hopf algebra can be used to define a composition law on a manifold that makes the manifold into a group under this composition law. The manifold will later become momentum space, so that we obtain in this way a composition law of momenta, and a corresponding momentum conservation law.

If the premises of prop. 31 are satisfied, the coproduct of the Hopf algebra H allows us to define a composition law for points on the base manifold M as follows. Given points $p, q \in M$ on the manifold, we define the components of $p \oplus q$ in the given chart as

$$(p \oplus q)_\mu \stackrel{\text{def}}{=} (\Delta\,(P_\mu))\,(p, q)\,. \tag{3.85}$$

Here we have written simply P_μ for its analog $T(P_\mu)$ in the representation, which we will continue to do. Clearly, by defining the components of $p \oplus q$ in the chart, we are defining $p \oplus q$ itself. Note, however, that if the coordinate functions map the chart on M to a proper subset of $\mathbb{R}^n$, the definition does not guarantee that $(p \oplus q)_\mu$ lies again inside the chart. This is something that has to be checked in each separate case but it will pose no problem in our analysis of the κ-Poincaré algebra later in this text.

3.4.3 Associativity of the Composition Law

Here we again identify points on the manifold with their vectors of components. We have

$$\begin{aligned}((p \oplus q) \oplus k)_\mu &= (\Delta\,(P_\mu))\,(p \oplus q, k) = (\Delta\,(P_\mu))\,((\Delta\,(P))\,(p, q)\,, k) && (3.86)\\ &= P_\mu^{(1)}\,(\Delta\,(P)\,(p, q))\,P_\mu^{(2)}(k) = P_\mu^{(1)}\,(\Delta\,(P))\,(p, q)\,P_\mu^{(2)}(k) && (3.87)\\ &= \left[P_\mu^{(1)}\,(\Delta\,(P)) \otimes P_\mu^{(2)}\right](p, q, k), && (3.88)\end{aligned}$$

where we used the Sweedler notation. Now by the Δ-analog of prop. 31, we can pull Δ into functions of generators, i.e., we can write

$$P_\mu^{(1)}\,(\Delta(P)) = \Delta(P_\mu^{(1)}(P)). \tag{3.89}$$

Using this equality and the coassociativity axiom we find that

$$\begin{aligned}((p \oplus q) \oplus k)_\mu &= \left[\Delta(P_\mu^{(1)}) \otimes P_\mu^{(2)}\right](p, q, k) = [(\Delta \otimes \mathrm{id}) \circ \Delta]\,(P_\mu)(p, q, k) && (3.90)\\ &= [(\mathrm{id} \otimes \Delta) \circ \Delta]\,(P_\mu)(p, q, k) = \left[P_\mu^{(1)} \otimes \Delta(P_\mu^{(2)})\right](p, q, k) && (3.91)\\ &= (p \oplus (q \oplus k))_\mu\,, && (3.92)\end{aligned}$$

and hence the composition law induced by the Hopf algebra is associative.

3.4.4 The Co-unit as Unit of the Composition Law

Next the co-unit of the Hopf algebra provides a unit element of the composition law via $\tilde{0}_\mu = \epsilon(P_\mu)$. Later this unit will be interpreted as the origin of momentum space. Note that we have

$$(p \oplus \tilde{0})_\mu = \Delta P_\mu (p, 0) = \Delta P_\mu (P, \epsilon(P)) = P_\mu^{(1)}(p) P_\mu^{(2)}(\epsilon(P)) \tag{3.93}$$

$$= P_\mu^{(1)}(p) \epsilon\left(P_\mu^{(2)}(P)\right) = P_\mu^{(1)}(p) \epsilon\left(P_\mu^{(2)}\right) = \left[P_\mu^{(1)} \epsilon\left(P_\mu^{(2)}\right)\right](p) \tag{3.94}$$

$$= [(1 \otimes \epsilon) \circ \Delta](P_\mu)(p) = \mathrm{id}(P_\mu)(p) = P_\mu(p) = p_\mu. \tag{3.95}$$

where we have used the co-unit axiom and the ϵ-analog of prop. 31. Similarly we find that $(\tilde{0} \oplus p) = p$. Hence the phase space origin $\tilde{0}$ is the identity with respect to the composition law $\oplus$.

3.4.5 Antipode Provides Inverse Elements

Having used Δ to provide the composition law, ϵ to provide the unit element, finally show that S provides the inverse of any given element in M under the composition law via

$$(\ominus p)_\mu \equiv (S(P_\mu))(p). \tag{3.96}$$

We will now prove that this definition implies that

$$p \oplus (\ominus p) = (\ominus p) \oplus p = \tilde{0}, \tag{3.97}$$

where $\tilde{0}_\mu = \epsilon(P_\mu)$ is the unit defined previously. We have

$$(p \oplus (\ominus p))^\mu = (\Delta P_\mu)(p, \ominus p) = P_\mu^{(1)}(p) P_\mu^{(2)}(\ominus p) = P_\mu^{(1)}(p) P_\mu^{(2)}(S(P)(p)), \tag{3.98}$$

where there is no Einstein summation. Writing $P_\mu^{(2)} = f(P)$ for the moment, we see that

$$P_\mu^{(2)}(S(P)(p)) = f(P)(S(P)(p)) \equiv (f \circ P)(S(P)(p)) \tag{3.99}$$

$$= f[P((S(P)(p)))] = f[(S(P)(p))] = (f \circ (S(P)))(p) \tag{3.100}$$

$$\equiv f(S(P))(p) = S(f(P))(p) = S\left(P_\mu^{(2)}\right)(p). \tag{3.101}$$

where if we have written $\equiv$ we have used the definition of how a function, like $f(P)$, acts on a points p on the manifold. Substituting this last relation into the former and using the antipode axiom, we obtain (again no Einstein summation)

$$\begin{aligned}
(p \oplus (\ominus p))^{\mu} &= P_{\mu}^{(1)}(p) S\left(P_{\mu}^{(2)}\right)(p) = \left[P_{\mu}^{(1)} S\left(P_{\mu}^{(2)}\right)\right](p) && (3.102)\\
&= \left[(\mu \circ (1 \otimes S))\left(P_{\mu}^{(1)} \otimes P_{\mu}^{(1)}\right)\right](p) && (3.103)\\
&= \left[(\mu \circ (1 \otimes S))\left(\Delta P_{\mu}\right)\right](p) = \left[\mu \circ (1 \otimes S) \circ \Delta\right](P_{\mu})(p) && (3.104)\\
&= (\eta \circ \epsilon)(P_{\mu})(p) = \eta\left(\epsilon(P_{\mu})\right)(p) = \epsilon(P_{\mu}) \cdot 1(p) = \epsilon(P_{\mu}) && (3.105)\\
&= \tilde{0}_{\mu}. && (3.106)
\end{aligned}$$

Here μ is the multiplication map $H \otimes H \to H$ and in the last line we used that the unit of the algebra is the map that sends all $p \mapsto 1$. The exact same procedure, but where instead of $1 \otimes S$, $S \otimes 1$ will appear, shows that we also have

$$(\ominus p) \oplus p = \tilde{0}. \tag{3.107}$$

Combining the results of the last couple of sections yields the following.

Proposition 32. *The composition law $(p, q) \mapsto p \oplus q$ induced by the Hopf algebra gives the manifold M the structure of a group. The identity element is given by $\tilde{0}$ and the inverse of an element p is given by $\ominus p$, as defined above.*

3.5 The κ-Poincaré Hopf Algebra

3.5.1 The Standard Basis

In section 3.3.2 we already encountered the algebra sector of the κ-Poincaré Hopf algebra as an example to illustrate nonlinear basis transformations, but here is where we officially introduce it, so we will repeat the formulas. In the standard basis in $1+1$ dimensions, the κ-Poincaré Hopf algebra sector is generated as an h-adic algebra by P_0, P_1, N with relations

$$[P_0, P_1] = 0, \qquad [N, P_0] = P_1, \qquad [N, P_1] = \kappa \sinh(P_0/\kappa). \tag{3.108}$$

The coproduct, co-unit and antipode are given on generators as

$$\Delta P_0 = P_0 \otimes 1 + 1 \otimes P_0, \tag{3.109}$$
$$\Delta P_1 = P_1 \otimes e^{P_0/2\kappa} + e^{-P_0/2\kappa} \otimes P_1, \tag{3.110}$$
$$\Delta N = N \otimes e^{P_0/2\kappa} + e^{-P_0/2\kappa} \otimes N, \tag{3.111}$$
$$\epsilon(P_0) = \epsilon(P_1) = \epsilon(N) = 0, \tag{3.112}$$
$$S(P_0) = -P_0, \tag{3.113}$$
$$S(P_1) = -e^{P_0/2\kappa} P_1 e^{-P_0/2\kappa}, \tag{3.114}$$
$$S(N) = -e^{P_0/2\kappa} N e^{-P_0/2\kappa}, \tag{3.115}$$

and by prop. 24 this is enough to specify the these maps on the entire algebra.

3.5.2 The Bicrossproduct Basis

Applying the basis transformation of section 3.3.2, we arive at the bicrossproduct basis, introduced in [13], which is the basis that is most often employed in physics applications. As we have already seen, the algebra sector in this basis reads

$$[P_0, \tilde{P}_1] = 0, \qquad [N, P_0] = P_1, \qquad [N, P_1] = \frac{\kappa}{2}\left(1 - e^{-2P_0/\kappa}\right) - \frac{1}{2\kappa} P_1^2. \tag{3.116}$$

For the coproduct, note that by Prop. 25, since P_0 is a primitive element, $\Delta\left(e^{P_0/2\kappa}\right) = e^{P_0/2\kappa} \otimes e^{P_0/2\kappa}$. Using this, and the fact that Δ is an algebra morphism we easily obtain the coproducts in the bicrossproduct basis,

$$\Delta(P_0) = P_0 \otimes 1 + 1 \otimes P_0, \tag{3.117}$$
$$\Delta(P_1) = P_1 \otimes 1 + e^{-P_0/\kappa} \otimes P_1, \tag{3.118}$$
$$\Delta(N) = N \otimes 1 + e^{-P_0/\kappa} \otimes N. \tag{3.119}$$

And similarly, using the fact that ϵ is an algebra morphism and S and algebra antimorphism, we obtain these maps in the bicrossproduct basis,

$$\epsilon(P_0) = \epsilon(P_1) = \epsilon(N) = 0, \tag{3.120}$$
$$S(P_0) = -P_0, \qquad S(P_1) = -e^{P_0/\kappa} P_1, \qquad S(N) = -e^{P_0/\kappa} N. \tag{3.121}$$

There exist a deformed version of the standard quadratic Casimir of the Poincaré algebra, which is referred to as the quadratic Casimir of the κ-Poincaré algebra. It is given in the bicrossproduct basis by

$$C = 4\kappa^2 \sinh^2\left(\frac{P_0}{2\kappa}\right) - (P_1)^2 e^{P_0/\kappa}. \tag{3.122}$$

Part II

The κ-Poincaré Model

Chapter 4

κ-Poincaré: Free Particles

The κ-Poincaré Hopf algebra introduced in the previous chapter has been the subject of a lot of research over the last couple of decades, often in connection with physics. There are compelling arguments suggesting that this algebra might emerge in certain limits of quantum gravity [14, 15, 16, 17][1]. The κ-Poincaré algebra is relevant in particular for the gravity-free, semi-classical limiting case $\hbar, G \to 0$ where the Planck mass $\sqrt{\hbar/G}$ (here and in the remainder of the thesis we set $c = 1$) retains a finite value. There is agreement on the fact that in this regime the Poincaré algebra should be somehow deformed. The hope is that in the future someone will be able to obtain the κ-Poincare algebra as the symmetry algebra of the above-mentioned $\hbar \to 0, G \to 0$ limit of some fundamental theory of quantum gravity. Assuming that this is indeed possible, it is most natural to expect that the deformation parameter κ would be given by some scalar multiple of the Planck mass, with the scaling factor being roughly of order 1. Today one therefore usually identifies κ simply with the Planck mass and this is what we will do throughout this thesis as well.
In the preceding chapter we have developed quite a bit of theory about Hopf algebras and in particular h-adic Hopf algebras like the κ-Poincaré Hopf algebra. We also already made a connection with physics here and there. In this chapter we will make this connection more explicit in the particular case of the κ-Poincaré algebra.

[1] Other authors argue that it is not the κ-Poincaré algebra that emerges but some other deformed algebra [18, 19, 20].

The starting point for the κ-Poincaré model is the κ-Poincaré Hopf algebra in the bicrossproduct basis, introduced in section 3.5.2. We will focus here on the description in $1+1$ dimensions, in which case the algebra sector reads

$$[P_0, P_1] = 0, \qquad [N, P_0] = P_1, \qquad [N, P_1] = \frac{\kappa}{2}\left(1 - e^{-2P_0/\kappa}\right) - \frac{1}{2\kappa}P_1^2, \tag{4.1}$$

with Casimir

$$C = 4\kappa^2 \sinh^2\left(\frac{P_0}{2\kappa}\right) - (P_1)^2 e^{P_0/\kappa}, \tag{4.2}$$

which is the only known Casimir element of the algebra in 1+1 dimensions. At this point only the algebra sector is relevant, and we will worry about the coalgebra sector later when we consider interactions. We will use the same method of obtaining the dynamics of this model as we did in section 1.3.2 for de Sitter spacetime. That is, we will impose that the algebra generators P_0, P_1, N be generators of a symmetry group that acts on the phase space of, at this stage, a single particle. Employing the machinery Hamiltonian mechanics, we may do this by representing the algebra on phase space and requiring that the Hamiltonian commute with the (represented) algebra generators. The most natural choice of Hamiltonian therefore is (the representation of) the mass Casimir. Our phase space will be the standard one, $\mathbb{R}^4$, consisting of elements of the form (x^0, x^1, p_0, p_1), with standard Poisson brackets,

$$\{x^\mu, p_\nu\} = \delta^\mu_\nu.$$

On this phase space the algebra can be represented by the functions

$$P_0 = p_0, \qquad P_1 = p_1, \qquad N = p_1 x^0 + x^1\left(\frac{\kappa}{2}\left(1 - e^{-2p_0/\kappa}\right) - \frac{(p_1)^2}{2\kappa}\right), \tag{4.3}$$

which satisfy

$$\{P_0, P_1\} = 0, \qquad \{N, P_0\} = P_1, \qquad \{N, P_1\} = \frac{\kappa}{2}\left(1 - e^{-2P_0/\kappa}\right) - \frac{1}{2\kappa}P_1^2. \tag{4.4}$$

This leads to the Hamiltonian

$$H = 4\kappa^2 \sinh^2\left(\frac{p_0}{2\kappa}\right) - (p_1)^2 e^{p_0/\kappa}. \tag{4.5}$$

Since we want the mass of a particle to be an invariant of the symmetry group as well, it is natural to postulate that the mass is also given by (the numerical value of) the mass Casimir, as it is in the undeformed case. This leads to the dispersion relation

$$4\kappa^2 \sinh^2\left(\frac{p_0}{2\kappa}\right) - (p_1)^2 e^{p_0/\kappa} = m^2. \tag{4.6}$$

Note that any scalar multiple[2] of the mass Casimir C is still a Casimir, but, of all these, only C reduces in the special relativistic limit, $\kappa \to \infty$, to the standard dispersion relation $m^2 = (p_0)^2 - (p_1)^2$. For a massless particle we have $m^2 = 0$ and the dispersion relation reduces to

$$p_1 = \pm\,\kappa\left(1 - e^{-p_0/\kappa}\right). \tag{4.8}$$

To find the worldines of particles, we use Hamilton's equations,

$$\dot{p}_\mu = \{p_\mu, H\} = 0, \quad \dot{x}^0 = \{x^0, H\} = 2\kappa \sinh\left(\frac{p_0}{\kappa}\right) - \frac{(p_1)^2 e^{p_0/\kappa}}{\kappa}, \tag{4.9}$$

$$\dot{x}^0 = \{x^1, H\} = -2p_1 e^{p_0/\kappa}. \tag{4.10}$$

The first equation shows that all momenta are constants of motion, and since

$$\frac{\mathrm{d}x^1}{\mathrm{d}x^0} = \frac{\dot{x}^1}{\dot{x}^0} = \frac{2\kappa p_1}{\kappa^2\left(e^{-\frac{2p_0}{\kappa}} - 1\right) + (p_1)^2} =: R_p \tag{4.11}$$

[2]In fact *any* function $f(C)$ of the (represented) Casimir is again an invariant, because for any symmetry generator G we have

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} f(C) = \{f(C), G\} = f'(C)\,\{C, G\} = f'(C) \cdot 0 = 0, \tag{4.7}$$

where λ is the group parameter corresponding to G. Hence the definition of mass in this context is somewhat ambiguous. Later we will derive, by postulating that momentum space has de Sitter geometry, a nontrivial such function f and hence an alternative definition of mass, which has the arguable advantage that the mass coincides with the rest energy of the particle, which here is not the case. These two definitions of mass, however, do always agree on whether a particle is massive or massless.

is again a constant on motion, the worldline of a particle with momentum p is given by a straight line with slope R_p,

$$x^1 = \bar{x}^1 + R_p x^0, \tag{4.12}$$

where $\bar{x}^1$ is an arbitrary constant, denoting the spatial position of the particle at vanishing x^0. For massless particles the slope R_p and the worldine conveniently reduce to

$$R_p = -\mathrm{sgn}(p_1)e^{p_0/\kappa}, \qquad x^1 = \bar{x}^1 - \mathrm{sgn}(p_1)e^{p_0/\kappa}x^0. \tag{4.13}$$

Here we have assumed that the energy p_0 is nonnegative, which we will continue to do throughout. This shows that the velocity of a massless particle of energy p_0 is not simply given by the 'speed of light' (which is 1 in our units), but by $e^{p_0/\kappa}$. This velocity is bounded from below by the speed of light (since $p_0 > 0$), but increases with increasing energy. Similarly, the energy dependence of the velocity of massive particles is deformed with respect to Special Relativity.

4.1 Intermezzo: Symmetries and Inertial Observers in Special Relativity

In order to fully appreciate the upcoming section 4.2 about observers in κ-Poincaré it is convenient to have the corresponding facts for Special Relativity near at hand. We briefly develop those facts here for the 1+1 dimensional case, working in a phase space setting.

Inertial observers in Special Relativity are defined via their symmetry group, which is the (identity component of the[3]) Poincaré group. In general, if we say that an observer A is related to an observer B by a (symmetry) transformation ϕ we mean that the the phase space coordinates of observer A are obtained from those of B by acting on them with ϕ. The (identity component of the) Poincaré group in 1+1 dimensions consists of the basic *translations* and *boosts* and any

[3]There are no observers for which the time direction or parity is reversed.

finite composition of those basic transformations. The action of translations over a^μ and boosts with rapidity ξ on the phase space of a particle is

$$T_a \begin{pmatrix} x^\mu \\ p_\mu \end{pmatrix} = \begin{pmatrix} x^\mu + a^\mu \\ p_\mu \end{pmatrix}, \qquad \Lambda_\xi \begin{pmatrix} x^\mu \\ p_\mu \end{pmatrix} = \begin{pmatrix} \Lambda^\mu{}_\nu \, x^\nu \\ \Lambda_\mu{}^\nu \, p_\nu \end{pmatrix}, \tag{4.14}$$

where

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \cosh\xi & \sinh\xi \\ \sinh\xi & \cosh\xi \end{pmatrix} \quad \text{and} \quad \Lambda_\mu{}^\nu = \left(\Lambda^{-1}\right)^\mu{}_\nu = \begin{pmatrix} \cosh\xi & -\sinh\xi \\ -\sinh\xi & \cosh\xi \end{pmatrix}. \tag{4.15}$$

We have used standard notation for the inverse of the Lorentz transformation. We need the *inverse* Lorentz transformation because we are dealing with the *covariant* (in contrast to contravariant) form of the momenta. (The above symmetries can be derived easily by computing the Killing vectors of Minkowski space, and exponentiating their action on the coordinates of the cotangent bundle, either to Minkowski spacetime or Minkowski momentum space. Since $T_a \circ T_b = T_{a+b}$ and $\Lambda_\xi \circ \Lambda_\zeta = \Lambda_{\xi+\zeta}$, and since the following 'commutation' relation holds for translations T_a over a^μ and boosts Λ_ξ of rapidity ξ,

$$T_a \circ \Lambda_\xi = \Lambda_\xi \circ T_{a'}, \qquad (a')^\mu = \Lambda^\mu{}_\nu a^\nu, \tag{4.16}$$

we can write any transformation between inertial observers as a composition of a single boost with a single translation. Thus two observers in Special Relativity are necessarily related by a transformation of the form

$$\Lambda_\xi \circ T_a. \tag{4.17}$$

There are two important observations we have to make here. First, notice that with respect to the action of these symmetries, spacetime and momentum space are completely decoupled, in the sense that a transformed spacetime coordinate depends only on the initial position and not the initial momentum, and that a transformed momentum depends only on the initial momentum and not the initial position. And moreover, the spacetime part of the transformation, say $x^\mu \to \Lambda^\mu{}_\nu x^\nu + a^\mu$, determines uniquely the transformation as a whole, including the momentum part. This justifies the fact that in Special Relativity we usually

relate different inertial observers by a *spacetime* transformation. We tend to take it for granted that this is possible, but in general it need not be the case. Indeed, we will see that observers in κ-Poincaré cannot be defined by a spacetime transformation alone; a complete phase space transformation has to be provided.
The second crucial property of these transformations is the fact that $(a')^\mu$ as in eq. (4.16) is again a *constant*, i.e., does not depend on x^μ or p_μ. If this were not the case, then $T_{a'}$ would not be an ordinary translation and we would therefore encounter additional transformations, like $\Lambda_{-\xi} \circ T_a \circ \Lambda_\xi = T_{a'}$, that might not be expressible in the form $\Lambda_\xi \circ T_a$. We will see that the latter is true for κ-Poincaré symmetry transformations.

4.2 Symmetries and Inertial Observers

4.2.1 Symmetries

The κ-Poincaré algebra, regarded as an algebra of infinitesimal transformations, can be integrated to yield finite phase space transformations. For translations this is trivial: since the translation generators are just the different components of momentum, which have canonical Poisson brackets with positions, a finite translation over a^μ is simply given by

$$p_\mu \to p_\mu, \qquad x^\mu \to x^\mu - a^\mu. \tag{4.18}$$

The finite form of the boosts was derived in [25] and used in [21] to construct the κ-Poincaré model. We here state the results in $1+1$D. By integrating the infinitesimal action on momenta (note that these do not involve spacetime positions at all),

$$p_0 \to p_0 + \xi\{N, p_0\} = p_0 + \xi p_1, \tag{4.19}$$

$$p_1 \to p_1 + \xi\{N, p_1\} = p_1 + \xi \left[\frac{\kappa}{2}\left(1 - e^{-2p_0/\kappa}\right) - \frac{1}{2\kappa}p_1^2\right], \tag{4.20}$$

it follows that a finite boost of rapidity ξ, written $x^\mu \to \tilde{x}^\mu, E \to \widetilde{E}$, $p \to \tilde{p}$, with $E \equiv p_0$ and $p \equiv p_1$ for readability, acts on momenta as

$$\widetilde{E} = E + \log\left[\left(\cosh(\xi/2) + \frac{p}{\kappa}\sinh(\xi/2)\right)^2 - e^{-2E/\kappa}\sinh^2(\xi/2)\right], \tag{4.21}$$

$$\tilde{p} = \kappa\left[\frac{\left(\cosh(\xi/2) + \frac{p}{\kappa}\sinh(\xi/2)\right)\left(\sinh(\xi/2) + \frac{p}{\kappa}\cosh(\xi/2)\right)}{\left(\cosh(\xi/2) + \frac{p}{\kappa}\sinh(\xi/2)\right)^2 - e^{-2E/\kappa}\sinh^2(\xi/2)} - \frac{e^{-2E/\kappa}\sinh(\xi/2)\cosh(\xi/2)}{\left(\cosh(\xi/2) + \frac{p}{\kappa}\sinh(\xi/2)\right)^2 - e^{-2E/\kappa}\sinh^2(\xi/2)}\right]. \tag{4.22}$$

The spacetime coordinates transform precisely as covectors to momentum space,

$$\tilde{x}^\mu = x^\nu \frac{\partial p_\nu}{\partial \tilde{p}_\mu}. \tag{4.23}$$

This indeed coincides infinitesimally with the infinitesimal action

$$x^0 \to x^0 + \xi\{N, x^0\} = x^0 - \xi e^{-2p_0/\kappa}x^1, \tag{4.24}$$

$$x^1 \to x^1 + \xi\{N, x^1\} = x^1 + \xi\left(\frac{p_1 x^1}{\kappa} - x^0\right). \tag{4.25}$$

In the limit $\kappa \to \infty$ all transformations reduce to those of Special Relativity (although with ξ replaced by $-\xi$, but this is merely the result of our conventions). We will not be concerned in this thesis with discrete symmetries like, for instance, time inversion. Therefore we define the κ-Poincaré *symmetry* group as the group generated by translations and boosts. Do not confuse this with what is usually called the κ-Poincaré group, which is a Hopf algebra dual to the κ-Poincaré Hopf algebra[4].

4.2.2 Inertial Observers

It is important to give a clear definition of what is meant by an inertial observer in the κ-Poincaré setting. And as in Special Relativity, the way to do this is by

[4]Sometimes the κ-Poincaré Hopf algebra is also referred to as the κ-Poincaré group. The terminology depends on the author.

defining the symmetry group relating inertial observers. Again, in general, if we say that an observer A is related to an observer B by a (symmetry) transformation ϕ we mean that the the phase space coordinates of observer A are obtained from those of B by acting on them with ϕ. The (minimal[5]) natural candidate for this symmetry group in the present case is the group consisting of (κ-) *translations* and (κ-) *boosts* and any finite composition of those basic transformations. Denoting these translations and boosts by T_a and Λ_ξ, respectively, it turns out that $T_a \circ T_b = T_{a+b}$ and $\Lambda_\xi \circ \Lambda_\zeta = \Lambda_{\xi+\zeta}$, just as is the case Special Relativity, but the 'commutation relation' (4.16) is now replaced by

$$T_a \circ \Lambda_\xi = \Lambda_\xi \circ T_{a'(p)}, \qquad a'(p)^\mu = \frac{\mathrm{d}\left(\Lambda_\xi\right)_\nu}{\mathrm{d}p_\mu}(p)a^\nu, \tag{4.26}$$

which is easily verified. (In the definition of $a'(p)$, Λ_ξ is understood to be the *momentum* part of the full phase space transformation.) To first order in ξ we can write $a'(p)$ as

$$a'(p)^0 = a^0 + \xi e^{-2p_0/\kappa}a^1, \tag{4.27}$$

$$a'(p)^1 = a^1 - \xi\left(\frac{p_1 a^1}{\kappa} - a^0\right). \tag{4.28}$$

We stress that writing $T_{a'(p)}$ here is, a priori, an abuse of notation, because $a'(p)$ is not a constant (it depends on p_μ), so $T_{a'(p)}$ is not an ordinary translation. Its action on the phase space of a particle is

$$T_{a'(p)}\left(x, p\right) = \left(x + a'(p), p\right), \tag{4.29}$$

which looks like a translation, but with a momentum-dependent translation parameter $a'(p)$. We will refer to these transformations as $a(p)$-translations or momentum-dependent translations. This kind of transformation is a symmetry of phase space. Actually, even in Special Relativity it is a symmetry of phase, but the crucial difference with the present case is that now the symmetry *actually relates inertial observers*, which in Special Relativity it does not, the reason being that in the present case the transformation can be written as a composition of our basic κ-translations and κ-boosts

$$T_{a'(p)} = \Lambda_{-\xi} \circ T_a \circ \Lambda_\xi, \tag{4.30}$$

[5]i.e., consisting of the fewest transformations.

in contrast to Special Relativity. In the limit $\kappa \to \infty$, we recover the known facts of Special Relativity, because in that case $a'(p)^\mu \to \Lambda^\mu{}_\nu \, a^\nu$, which is a constant four-vector.

An essential difference with Special Relativity (section 4.1) is that κ-Poincaré observers may be related by symmetries for which spacetime and momentum space do not decouple: the action of $T_{a'(p)}$ on x depends on the accompanying momentum p. Hence, contrary to what is the case in Special Relativity, we cannot identify observers in κ-Poincaré by a spacetime transformation alone. A full phase space transformation is required. This is in essence the reason that the theory features Relative Locality effects (see section 2).

4.3 Compatibility with Relative Locality

In this section we discuss the compatibility of the free-particle κ-Poincaré model with Relative Locality (section 2). Although an important part of the Relative Locality Framework (RLF) is its description of interactions, here we are discussing only free-particle systems, so we will ignore the interaction part (and return to it in the subsequent chapters). We will see that the κ-Poincaré model is indeed compatible with the RLF, up to a redefinition of mass (which poses no problem at all), at least when interactions are disregarded. In the next chapter we will find, however, that this will not be the case (at least not in a covariant way) when interactions are taken into account.

To show that the single-particle κ-Poincaré model is compatible with the RLF we need to find a momentum space geometry that reproduces, via the equations of the RLF, all equations of the κ-Poincaré model. This geometry turns out to be de Sitter. This was derived in [21], and in section 4.3.1 we will show the reasoning behind it. At this point we will simply observer that it works.
In chapter section 2.3 the de Sitter Relative Locality model (dS-RL) has already been discussed. Hence our only task at this points is to note that the two free-particle models, although derived in a completely different way, are in fact identical. Well, not precisely identical; the only difference is in their definition

of the mass of a particle,

$$\kappa\text{-Poincaré}: \qquad m^2 = 4\kappa^2 \sinh^2\left(\frac{p_0}{2\kappa}\right) - (p_1)^2 e^{p_0/\kappa}, \tag{4.31}$$

$$\text{dS Relative Locality}: \qquad \tilde{m} = \kappa \operatorname{arccosh}\left(\cosh\frac{p_0}{\kappa} - e^{\frac{p_0}{\kappa}}\frac{(p_1)^2}{2\kappa^2}\right), \tag{4.32}$$

where we have denoted the κ-Poincaré mass by m and the dS-RL mass by m'. It can be checked easily that

$$\tilde{m} = \kappa \operatorname{arccosh}\left(\frac{m^2}{2\kappa^2} + 1\right), \tag{4.33}$$

so there is a one-to-one relationship between a given m and a given $\tilde{m}$ (at least assuming they are both non-negative, which we will always do). Both definitions have the correct special relativistic limit, and $m = 0$ if and only if $\tilde{m} = 0$. The arguable advantage of the dS-RL mass is that $\tilde{m}$ coincides with the rest mass of the particle, while m does not. Of course, the same interpretation can be achieved in the κ-Poincaré model by simply redefining the mass of a particle according to the formula above. Hence we may say that the two models are equivalent, if not identical. The κ-Poincaré model for free particles is therefore completely compatible with the RLF as a κ-Poincaré-invariant theory[6].

The Reason that the Two Models Coincide

If it seems mysterious why the two models, using completely different methods, arrive at the same theory, we will provide here, for completeness, the – admittedly not particularly insightful – mathematical reason. In both approaches the momenta are trivially constant, and the equation of motion for x^μ, in the geometric approach is

$$\dot{x}^\mu = -N\frac{\partial D^2(p)}{\partial p_\mu}, \tag{4.34}$$

[6] The careful reader might remember from chapter 2 that the translations, as we have defined them in the present chapter, are not symmetries of the full RLF, and may therefore wonder why we are stating here that the model is compatible with the RLF as a κ-Poincaré-invariant theory. The point is that as long as we ignore interactions, which we are doing here, these translations *are* in fact a symmetry of the RLF, because the only reason that they would not be is the presence of interaction vertices.

with N some Lagrange multiplier and $D(p)$ the geodesic distance from p to the origin of momentum space. The point is that this geodesic distance $D(p)$ can be written as a function of the Casimir C, namely

$$D = \kappa \operatorname{arccosh}\left(\frac{C}{2\kappa^2} + 1\right), \tag{4.35}$$

and hence we can write (4.34) as

$$\dot{x}^\mu = -N\frac{\partial D^2(p)}{\partial p_\mu} = -2N\frac{\partial D(p)}{\partial p_\mu} = -2N\frac{\partial D(p)}{\partial C}\frac{\partial C}{\partial p_\mu} = -2N\frac{\partial D(p)}{\partial C}\{x^\mu, C\}. \tag{4.36}$$

This shows that the quotient $\dot{x}^1/\dot{x}^0$ is given by the same expression in the two approaches, and therefore the worldlines are the same.

4.3.1 κ-Poincaré Momentum Space

At this point it is convenient to give a clear description of the momentum space corresponding to the κ-Poincaré model. We have just seen that the model coincides with the RLF model when one postulates that momentum space has de Sitter geometry with dS radius κ. But even without referring to the RLF, we can associate to the κ-Poincaré model a dS momentum space, following [21]. First of all, one can check that the κ-Poincaré boosts (4.21) leave the comoving dS momentum space metric $\mathrm{d}s^2 = \mathrm{d}E^2 - e^{2p_0/\kappa}\mathrm{d}p^2$ invariant. But there is a more constructive derivation. Since we have identified the κ-Poincaré Hopf algebra generators with coordinate functions of the momentum space manifold, a change of coordinates corresponds to a (nonlinear) basis change of the Hopf algebra. With this in mind, let us apply the transformation that we also used in chapter 1 relating dS comoving coordinates and Minkowski embedding coordinates, given by

$$\eta_0 = \kappa \sinh\left(\frac{E}{\kappa}\right) + \frac{e^{E/\kappa}P^2}{2\kappa}, \tag{4.37}$$

$$\eta_1 = e^{E/\kappa}P, \tag{4.38}$$

$$\eta_2 = \kappa \cosh\left(\frac{E}{\kappa}\right) - \frac{e^{E/\kappa}P^2}{2\kappa}. \tag{4.39}$$

In the new basis according to this nonlinear change, the algebra sector of the κ-Poincaré Hopf algebra reads

$$[\eta_0, \eta_1] = 0, \qquad [N, \eta_0] = \eta_1, \qquad [N, \eta_1] = \eta_0, \tag{4.40}$$

which we recognize as the undeformed Poincaré algebra. This suggests that the coordinate functions η_μ corresponding to these generators should be the coordinate functions of flat Minkowski space, although with the constraint that $-\eta_0^2 + \eta_1^2 + \eta_2^2 = \kappa^2$. This is precisely the definition of the dS submanifold of Minkowski space. Transforming back to the basis corresponding to the generators E and p, we may now interpret this nonlinear change as a coordinate transformation on the momentum space manifold, leading us naturally to comoving coordinates as a preferred coordinate system on dS momentum space, just like Minkowski coordinates constitute a preferred coordinate system on Minkowski space.
Not all of dS space is covered by physical momenta, however. First of all, according to the preceding derivation, the momenta lie only the comoving chart. In Minkowski embedding coordinates this chart consists of the points (η_0, η_1, η_2) which satisfy $\eta_0 + \eta_2 > 0$. But there are two additional restrictions. For instance, we require particles to have nonnegative energies, $p_0 \geq 0$. In embedding coordinates this means that

$$\log\left(\frac{\eta_0 + \eta_2}{\kappa}\right) \geq 0, \qquad \text{i.e.,} \qquad \eta_0 + \eta_2 \geq \kappa. \tag{4.41}$$

Also, we only allow particles with nonnegative mass[7]. Since

$$\tilde{m} = \kappa \operatorname{arccosh}\left(\cosh\frac{p_0}{\kappa} - e^{\frac{p_0}{\kappa}}\frac{(p_1)^2}{2\kappa^2}\right) = \kappa \operatorname{arccosh}\left(\frac{\eta_2}{\kappa}\right), \tag{4.42}$$

this requires that $\eta_2 \geq \kappa$. Hence in terms of embedding coordinates, the momentum space of the κ-Poincaré model is nicely summarized by fig. 4.1.

[7]We stress again that $\tilde{m} \geq 0$ is equivalent to $m \geq 0$, so that it does not matter in this regard which definition of mass we use.

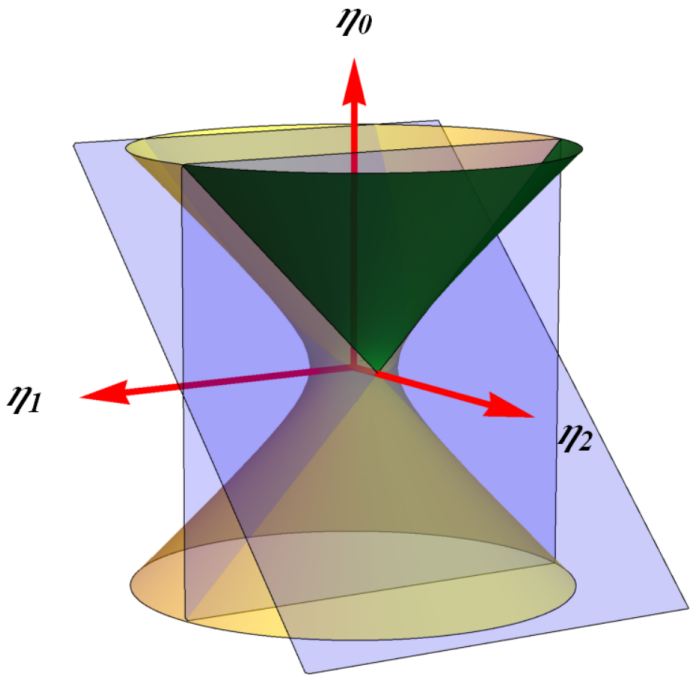


Figure 4.1: κ-Poincaré momentum space in 1+1 dimensions, embedded in 2+1 dimensional Minkowski space. The hyperboloid consisting of the green and yellow patches represents the entire dS space. The part of dS space above the diagonal blue ($\eta_0 + \eta_2 = \kappa$) plane is the part of dS space which is described by comoving coordinates (p_0, p_1). And the part of this which, additionally, lies in 'front' of the vertical blue ($\eta_2 = 0$) plane corresponds to momenta with a *real* (and automatically nonnegative) mass; this part, the actual κ-Poincaré momentum space, is colored green.

4.4 Phenomenology

4.4.1 Lateshift

We are now in the position to describe an interesting effect in de Sitter momentum space, comparable to, and in a sense dual to, the redshift effect for de Sitter *spacetime* discussed in 1.3.3. Where in that case we considered two photons emitted at different times with the same energy, we now consider two photons emitted at the same time with different energies. First let us write down the action of a^μ-translations on phase space. Note that phase space now has the standard Poisson structure, without the extra minus, and this gives an overall sign difference with the dS spacetime formulas (at least in the Special Relativistic limits, where the two models should yield the same formulas). So it seems most natural to add an overall minus sign to the definition translation parameter a^μ in the present analysis. This does not change any physics, of course, but it makes it easier to compare the formulas. The translations then act as follows

$$x^0 \to x^0 - a^0, \tag{4.43}$$
$$x^1 \to x^1 - a^1, \tag{4.44}$$
$$p_0 \to p_0, \tag{4.45}$$
$$p_1 \to p_1. \tag{4.46}$$

We again consider two observers, Alice and Bob. Alice emits both photons from her spacetime origin, with energies p^0 and $\tilde{p}^0$. For the first photon, the worldline is simply

$$x^1_A = e^{p_0/\kappa} x^0_A, \tag{4.47}$$

given that it moves to the 'right'. Bob is defined in a similar way as in section 1.3.3, by a translation with respect to Alice in such a way that the first photon crosses his spacetime origin. Using the action of the translations we find that this is the case if and only if the translation parameters satisfy

$$a^1 = e^{p_0/\kappa} a^0. \tag{4.48}$$

The photon's worldline will then be described by Bob as

$$x^1_B = e^{p_0/\kappa} x^0_B \tag{4.49}$$

and it is easy to see that the time, according to Bob, that the first photon was emitted by Alice and the time it was received by himself are given by

$$x^0_{B@A} = -a^0, \qquad x^0_{B@B} = 0. \tag{4.50}$$

Here we use again the notation with subscripts (or superscripts) like $B@A$, which means that the given quantity is evaluated by Bob, "B", at Alices spatial origin, "$@A$". The second photon will be described by Bob as traveling along the worldline

$$\tilde{x}^1_B = e^{\tilde{p}_0/\kappa} x^0_B + a^0 \left(e^{\tilde{p}_0/\kappa} - e^{p_0/\kappa} \right), \tag{4.51}$$

so the time at which Bob measures it (i.e., when $x^1_B = 0$) is given by

$$\tilde{x}^0_{B@B} = -a^0 \left(1 - e^{(p_0 - \tilde{p}_0)/\kappa} \right) \tag{4.52}$$

and the time, according to Bob, that Alice emitted it (i.e., when $\tilde{x}^1_B = -a^1$) is

$$\tilde{x}^0_{B@A} = -a^0. \tag{4.53}$$

Hence if we define the travel times according to Bob as

$$\Delta_B \tilde{x}^0 = \tilde{x}^0_{B@B} - \tilde{x}^0_{B@A}, \qquad \Delta_B x^0 = x^0_{B@B} - x^0_{B@A}, \tag{4.54}$$

then we obtain the formula

$$\Delta_B \tilde{x}^0 = e^{-\left(\tilde{p}_0^{A@A} - p_0^{A@A}\right)/\kappa} \Delta_B x^0. \tag{4.55}$$

Comparing this to the previously obtained formula (1.78),

$$\tilde{p}_0^{B@B} = e^{-H\left(\Delta_A \tilde{x}^0 - \Delta_A x^0\right)} p_0^{B@B}, \tag{4.56}$$

for de Sitter *spacetime*, we notice a very interesting duality between the two cases! The two formulas are basically the same, but with the roles of p_0 and Δx^0 interchanged. This was first discovered in the paper [33] (see also [34]), where this new effect, dual to redshift, was dubbed *lateshift*.

Chapter 5

κ-Poincaré with a Single Interaction

5.1 The Deformed Momentum Conservation Law

As explained in section 3.4.2, the κ-Poincaré Hopf algebra determines a momentum composition rule according to

$$(p \oplus q)_\mu = (\Delta(P_\mu))(p, q), \tag{5.1}$$

where the relevant coproducts $\Delta(P_\mu)$ are given by equations (3.117),(3.118),

$$\Delta(P_0) = P_0 \otimes 1 + 1 \otimes P_0, \tag{5.2}$$
$$\Delta(P_1) = P_1 \otimes 1 + e^{-P_0/\kappa} \otimes P_1. \tag{5.3}$$

The momentum composition rule thus becomes

$$(p \oplus q)_0 = p_0 + q_0, \qquad (p \oplus q)_1 = p_1 + e^{-p_0/\kappa} q_1. \tag{5.4}$$

This composition rule will be interpreted as a deformed momentum conservation law. In e.g. the interaction $p, q \to k$ the total momentum, calculated with $\oplus$, should be conserved:

$$p \oplus q \stackrel{!}{=} k. \tag{5.5}$$

Note that $\oplus$ is associative (see section 3.4.3) but not commutative, so it does matter in which order we add the momenta. There are different proposals in the literature on how to cope with this fact. One is the postulate that particles really can interact with different relative 'orderings', corresponding to different total momenta; another is that one should use a symmetrized composition law:

$$p \oplus_{\text{symm}} q = \frac{1}{2}(p \oplus q + q \oplus p) = q \oplus_{\text{symm}} p. \tag{5.6}$$

In this text we will mainly stick to the former proposal.

Consistency check

An important consistency check should be performed at this point, relating the dispersion relation (4.6) with the momentum composition rule (5.4). Namely, we need to make sure that if p, q are the momenta of two physical particles then $p \oplus q$ is again the momentum of a physical particle, that is, $(p \oplus q)_0$ must be again nonnegative and there must be some mass for which $p \oplus q$ satisfies the dispersion relation. The first requirement is trivially satisfied. And looking at the dispersion relation, we immediately see that the statement that there is some mass for which a given momentum p satisfies the dispersion relation, is equivalent to the one that

$$4\kappa^2 \sinh^2\left(\frac{p_0}{2\kappa}\right) - (p_1)^2 e^{p_0/\kappa} \geq 0, \tag{5.7}$$

which in turn is equivalent to

$$|p_1| \leq \kappa\left(1 - e^{-p_0/\kappa}\right). \tag{5.8}$$

Now suppose that the momenta p and q satisfy this relation and let $k = p \oplus q$. Then we easily see that

$$|k_1| = |p_1 + e^{-p_0/\kappa} q_1| \leq |p_1| + e^{-p_0/\kappa}|q_1| \leq \kappa\left(1 - e^{-p_0/\kappa}\right) + e^{-p_0/\kappa}\kappa\left(1 - e^{-q_0/\kappa}\right) \tag{5.9}$$

$$= \kappa\left(1 - e^{-(p_0+q_0)/\kappa}\right) = \kappa\left(1 - e^{-k_0/\kappa}\right) \tag{5.10}$$

and hence $p \oplus q$ is indeed again the momentum of a physical particle. In other words, the physical momentum space is closed under the composition law.

Group Structure

As discussed in section 3.4 the momentum composition law induced by the κ-Poincaré Hopf algebra makes momentum space into a group. The unit element is given in terms of the co-unit of the Hopf algebra in the bicrossproduct basis as

$$\tilde{0}_\mu = \epsilon(P_\mu) = 0. \tag{5.11}$$

This momentum will be called the *origin* of momentum space. Thus the origin of momentum space is simply the point that has vanishing comoving coordinates. Similarly the inverse of a momentum p_μ is provided by the antipode of the Hopf algebra via

$$(\ominus p)_\mu \equiv (S(P_\mu))(p), \tag{5.12}$$

which leads to

$$(\ominus p)_0 = -p_0, \qquad (\ominus p)_1 = -e^{p_0/\kappa} p_1. \tag{5.13}$$

One easily checks that we indeed have

$$p \oplus (\ominus p) = (\ominus p) \oplus p = \tilde{0}.$$

5.2 Symmetries

By the very construction of the theory, the translations and boosts are symmetries of the free-particle κ-Poincaré model. When interactions are present, however, we must also check if the symmetries leave the momentum conservation law invariant. The usual requirement is that if $p \oplus q = k$ then $\phi(p) \oplus \phi(q) = \phi(k)$ for any symmetry transformation ϕ. We will find, however, that we need to generalize this requirement, because it is in fact a little too restrictive and does not hold for boosts, while a generalized version does. For translations it is a trivial exercise to see that that the requirement does hold because the translations do not affect the momenta, i.e., for translations $\phi(p) = p$.

5.2.1 Boosts and the Backreaction

To see that the usual covariance requirement ($p \oplus q = k \Rightarrow \Lambda_\xi(p) \oplus \Lambda_\xi(q) = \Lambda_\xi(k)$, denoting a boost of rapidity ξ by Λ_ξ) does not hold for the deformed boosts it suffices to consider two massless particles, the spatial momenta of which we may express as

$$p_1 = \kappa \left(1 - e^{-p_0/\kappa}\right), \tag{5.14}$$

$$q_1 = \kappa \left(1 - e^{-q_0/\kappa}\right) \tag{5.15}$$

in terms of their energies. Then it follows that the composed momentum is also massless, $(p \oplus q)_1 = \kappa \left(1 - e^{-(p \oplus q)_0/\kappa}\right)$, and the boosts simplify enormously. We compute that

$$\left(\Lambda_\xi(p \oplus q)\right)_1 = \kappa - \frac{\kappa}{e^\xi \left(e^{\frac{p_0+q_0}{\kappa}} - 1\right) + 1}, \tag{5.16}$$

whereas

$$\left(\Lambda_\xi(p) \oplus \Lambda_\xi(q)\right)_1 = \kappa - \frac{\kappa}{\left(e^\xi \left(e^{p_0/\kappa} - 1\right) + 1\right)\left(e^\xi \left(e^{q_0/\kappa} - 1\right) + 1\right)}, \tag{5.17}$$

which is clearly not for all p_0, q_0 equal to the former expression. Hence the usual covariance condition is not satisfied. However, it was shown in [35], and applied in the present setting in [21], that a generalized covariance condition holds for the κ-Poincaré conservation law under boosts. In this generalization, different momenta are not necessarily boosted with the same rapidity. Although one cannot guarantee that

$$p \oplus q = k \qquad \Rightarrow \qquad \Lambda_\xi(p) \oplus \Lambda_\xi(q) = \Lambda_\xi(k), \tag{5.18}$$

it turns out one *can* guarantee that

$$p \oplus q = k \qquad \Rightarrow \qquad \Lambda_\xi(p) \oplus \Lambda_{\xi'}(q) = \Lambda_\xi(k), \tag{5.19}$$

provided that ξ', the rapidity of the *second incoming particle* of a vertex, is a suitable function of ξ and p. This then renders the model invariant under boosts.

The functional form of ξ' turns out to be

$$\xi' = \xi \triangleleft p \equiv 2\text{arcsinh}\left(\frac{\sinh\left(\frac{\xi}{2}\right) e^{-\frac{p_0}{\kappa}}}{\sqrt{\left(\cosh\left(\frac{\xi}{2}\right) + \frac{p_1 \sinh\left(\frac{\xi}{2}\right)}{\kappa}\right)^2 - \sinh^2\left(\frac{\xi}{2}\right) e^{-\frac{2p_0}{\kappa}}}}\right). \quad (5.20)$$

This expression is always well-defined, i.e., the argument of the square root is always positive and nonzero, but we postpone the proof of this until we have the expression in embedding coordinates (section 5.4.2). The conservation law is indeed invariant when boosts are applied to interacting particles in this way. The map $(\xi, p) \mapsto \xi' = \xi \triangleleft p$ is called the backreaction of p on ξ. One also says that the particle with momentum q 'gets a backreaction' form the particle with momentum p. Next let us see what this implies for more general interactions. Consider the process $p, q, \ell \to k$. We want to know how each individual particle in this interaction needs to transform under a boost (that is, with what rapidity), given that the first[1] incoming particle (which we take by convention to be p) transforms with rapidity ξ. Treating $q \oplus \ell$ as a single momentum for the moment, we know, given the above conclusion, that the conservation law $p \oplus q \oplus \ell = k$ is invariant if $q \oplus \ell$ transforms with rapidity $\xi \triangleleft p$, while p and k transform with ξ. By the same logic, if $q \oplus \ell$, as a single momentum, transforms with $\xi \triangleleft p$, then q should transform with $\xi \triangleleft p$ and ℓ should transform with $(\xi \triangleleft p) \triangleleft q$. In conclusion, if $p \oplus q \oplus \ell = k$ then it follows that

$$\Lambda_\xi(p) \oplus \Lambda_{\xi \triangleleft p}(q) \oplus \Lambda_{(\xi \triangleleft p) \triangleleft q}(\ell) = \Lambda_\xi(k) \quad (5.21)$$

and hence this ensures that also the reaction $p, q, l \to k$ behaves covariantly under boosts, provided the momenta transform as indicated. This is easily generalized to arbitrary interactions $p^{(1)} + \cdots + p^{(N)} \to k^{(1)} + \cdots + k^{(M)}$: each $p^{(i)}$ should transform with rapidity $\xi_i = \xi \triangleleft p^{(1)} \triangleleft \cdots \triangleleft p^{(i-1)}$, and each $k^{(i)}$ should transform with rapidity $\xi'_i = \xi \triangleleft k^{(1)} \triangleleft \cdots \triangleleft k^{(i-1)}$, i.e., a given incoming (outgoing) particle gets a backreaction from all incoming (outgoing) particles that come before it in the order of composition. The rapidity ξ with which the first incoming particle

[1] By convention, we take the order of the incoming/outgoing particles as that in which they are written down in an expression like $p + q + \ell \to k$.

and the first outgoing particle transform, and which determines all of the other rapidities, is called the *bare rapidity*. This guarantees that the conservation law is invariant: if $\tilde{p}^{(i)} = \Lambda_{\xi_i}\left(p^{(i)}\right)$ and $\tilde{k}^{(i)} = \Lambda_{\xi'_i}\left(k^{(i)}\right)$ then we have the equivalence

$$p^{(1)} \oplus \cdots \oplus p^{(N)} = k^{(1)} \oplus \cdots \oplus k^{(M)} \quad \Leftrightarrow \quad \tilde{p}^{(1)} \oplus \cdots \oplus \tilde{p}^{(N)} = \tilde{k}^{(1)} \oplus \cdots \oplus \tilde{k}^{(M)}. \tag{5.22}$$

This modified action of boosts on interacting particles renders the theory invariant under boosts, and hence under the whole κ-Poincaré transformation group, at least when one considers only particles that can interact at most once in there lifetimes. We will see in the next chapter, that this method is not sufficient to deal with particles that undergo multiple interactions. For that case, however, we have developed an extended version of this single-interaction 'backreaction method', which does succeed in making the momentum composition law (and hence the entire model) behave covariantly for a large class of scenarios with multiple interactions. This is our main new result.

The Backreaction Is a Right Action

Here we prove that the backreaction $(\xi, p) \mapsto p \triangleleft \xi$, as introduced above, is a right action of the group of momenta on the group of boosts or, equivalently, on the rapidities. We have seen that

$$\Lambda_\xi(p \oplus q) = \Lambda_\xi(p) \oplus \Lambda_{\xi \triangleleft p}(q) \tag{5.23}$$

for all p, q. Now we apply this identity in two ways to $p \oplus q \oplus k$, the associativity of the composition law ensuring that the two expressions are equal:

$$\Lambda_\xi(p \oplus (q \oplus k)) = \Lambda_\xi(p) \oplus \Lambda_{\xi \triangleleft p}(q \oplus k) = \Lambda_\xi(p) \oplus \Lambda_{\xi \triangleleft p}(q) \oplus \Lambda_{(\xi \triangleleft p) \triangleleft q}(k), \tag{5.24}$$

$$\Lambda_\xi((p \oplus q) \oplus k) = \Lambda_\xi(p \oplus q) \oplus \Lambda_{\xi \triangleleft (p \oplus q)}(k) = \Lambda_\xi(p) \oplus \Lambda_{\xi \triangleleft p}(q) \oplus \Lambda_{\xi \triangleleft (p \oplus q)}(k). \tag{5.25}$$

Using the fact that momenta form a group under $\oplus$, and that the expressions must be equal for all k, this is equivalent to

$$\Lambda_{(\xi \triangleleft p) \triangleleft q} = \Lambda_{\xi \triangleleft (p \oplus q)}, \tag{5.26}$$

which, since boosts of different rapidities are not equal[2], implies that

$$(\xi \triangleleft p) \triangleleft q = \xi \triangleleft (p \oplus q). \tag{5.27}$$

Also one quickly verifies that $\xi \triangleleft 0 = \xi$, so that $\triangleleft$ is indeed a right action.

5.3 Compatibility with Relative Locality

5.3.1 Translations

As discussed in section 2.2, translations should be implemented in the Relative Locality framework (RLF) by translating the z^μ coordinates, instead of the x^μ coordinates in order for the theory to be translation invariant. These alternative translations are given by

$$z^\mu \to z^\mu + a^\mu, \qquad \tilde{x}^\mu_I = x^\mu_I \mp a^\nu \frac{\partial \mathscr{K}_\nu}{\partial p^I_\mu}(p^I), \tag{5.28}$$

where the upper sign corresponds to outgoing particles in the z^μ interaction vertex and the lower sign to incoming particles. Of course, these translations are still a symmetry of the κ-Poincaré model as well, since they do not affect the momenta (and hence the dispersion relation and the conservation law) and only change the spacetime coordinates by a time-independent factor, leaving the equation of motion invariant. With these translations the κ-Poincaré model becomes compatible with the RLF as a translation-invariant theory. It is a consequence of the fact that the two were compatible in the noninteracting (chapter 4) case and the fact, discussed in section 2.2, that the additional 'interaction' equation,

$$x^\mu_I(\lambda_0) = \mp z^\nu \frac{\partial \mathscr{K}_\nu}{\partial p^I_\mu}\bigg|_{\lambda=\lambda_0}, \tag{5.29}$$

coming from the RLF, is also left invariant by these translations.

[2]This can be checked for instance by looking at massless particles, for which the boosts simplify enormously

5.3.2 Boosts

For boosts the situation is a bit more complicated, and it turns out that the κ-Poincaré model is *not* compatible with the RLF as a boost-invariant theory. The reason is that the interaction equation (5.29) cannot be made boost-invariant. One might hope to define a suitable transformation behavior of z^μ under boosts[3] such that the equation would be left invariant, but this is not possible. Let us see why. The required transformation of z^μ is found by writing down eq. (5.29) in the boosted system, substituting the known transformations of x^μ and p_μ, and solving for the transformed z^μ. The result is that z^μ should transform according to

$$z \to \tilde{z} \stackrel{!}{=} z \frac{\partial \mathscr{K}}{\partial p^I} \frac{\partial p^I}{\partial \tilde{p}^I} \left(\frac{\partial \tilde{\mathscr{K}}}{\partial \tilde{p}^I} \right)^{-1}, \tag{5.30}$$

written in matrix notation. We can understand immediately why this leads to trouble. Eq. (5.30) has to hold for each particle I in the interaction. Hence all the different expressions for $\tilde{z}$ would have to conspire in some way in order for the transformation to be well-defined. For the κ-Poincaré model this does not happen, and the reason for this turns out to be the nontriviality of the backreaction (section 5.2.1). One can prove, but we leave this to the reader, that when the backreaction is trivial (that is, absent) then the transformation of z^μ can be simplified to

$$\tilde{z} = z \frac{\partial \mathscr{K}}{\partial \tilde{\mathscr{K}}}, \tag{5.31}$$

which does not depend anymore on the particle. But when the backreaction is nontrivial this is not the case, and indeed in the case of the κ-Poincaré model the transformation (5.30) is ill-defined. In chapter 7 we will modify the momentum composition law, so that it does not need a backreaction, and hence in that case the transformation of z^μ will in fact be well-defined.

[3] Only the boost behavior of x^μ and p_μ is defined at this point, not that of z^μ.

5.4 Description in terms of Minkowski Embedding Coordinates

Here we reformulate the κ-Poincaré model in terms of Minkowski embedding coordinates (on momentum space) instead of comoving coordinates. This can be very useful in certain situations. One important remark, though, is that in embedding coordinates the limit $\kappa \to \infty$ is ill-defined and makes no sense[4]. This is basically because the coordinate transformation to embedding coordinates is only defined for finite κ. This is important to keep in mind; if we are interested in the Special Relativistic limit we should always first go back to comoving coordinates and only then take the limit.

5.4.1 Boosts

As explained in chapter 1, comoving coordinates and Minkowski embedding coordinates of de Sitter space are related, in $1+1$ (dS) dimensions, by the embedding

$$\eta_0 = \kappa \sinh E/\kappa + e^{E/\kappa}\frac{p^2}{2\kappa}, \tag{5.32}$$

$$\eta_1 = e^{E/\kappa}p, \tag{5.33}$$

$$\eta_2 = \kappa \cosh E/\kappa - e^{E/\kappa}\frac{p^2}{2\kappa}, \tag{5.34}$$

the image of which consists of those (η_0, η_1, η_2) that satisfy $\eta_1 + \eta_2 > 0$. The inverse map is given by

$$E = \kappa \ln\left(\frac{\eta_0 + \eta_2}{\kappa}\right), \qquad p = \frac{\kappa\,\eta_1}{\eta_0 + \eta_2}. \tag{5.35}$$

It turns out that the κ-Poincaré boost of rapidity ξ coincides precisely with the usual Lorentz boost of rapidity ξ of the ambient Minkowski space,

$$\tilde{\eta}_0 = \cosh(\xi)\eta_0 + \sinh(\xi)\eta_1, \quad \tilde{\eta}_1 = \sinh(\xi)\eta_0 + \cosh(\xi)\eta_1, \quad \tilde{\eta}_2 = \eta_2, \tag{5.36}$$

[4]Unless at the same time one sends $\eta_2 \to \infty$.

i.e., the κ-Poincaré boosts are in essence simply Lorentz boosts. This is easily generalized to higher dimensions. The κ-Poincaré boost in the i-direction will correspond to the Minkowski boost in the i-direction, and the κ-Poincaré rotation (which is undeformed) in the ij-plane will correspond to the same rotation in Minkowski space.

5.4.2 Composition Law and Backreaction

The κ-Poincaré composition law of momenta (eq. (5.4)) is given in embedding coordinates by

$$(\eta \oplus \zeta)_0 = \frac{\zeta_0\kappa + \zeta_1\eta_1}{\eta_0 + \eta_2} + \frac{\eta_0(\zeta_0 + \zeta_2)}{\kappa}, \tag{5.37}$$

$$(\eta \oplus \zeta)_1 = \frac{\eta_1(\zeta_0 + \zeta_2)}{\kappa} + \zeta_1, \tag{5.38}$$

$$(\eta \oplus \zeta)_4 = -\frac{\zeta_0\kappa + \zeta_1\eta_1}{\eta_0 + \eta_2} - \frac{\eta_0(\zeta_0 + \zeta_2)}{\kappa} + \frac{(\eta_0 + \eta_2)(\zeta_0 + \zeta_2)}{\kappa} \tag{5.39}$$

and the backreaction is given by

$$\xi \triangleleft \eta = 2\text{arcsinh}\left(\frac{\kappa \sinh\left(\frac{\xi}{2}\right)}{\sqrt{(\eta_0 + \eta_2)(\eta_0 \cosh(\xi) + \eta_1 \sinh(\xi) + \eta_2)}}\right). \tag{5.40}$$

In this form it is easy to see that the backreaction is always well-defined, i.e., that the argument of the square root is always positive and nonzero for any momentum in the comoving chart. We have $\eta_0 + \eta_2 = \kappa\, e^{p_0/\kappa} > 0$ and and $\eta_0 \cosh(\xi) + \eta_1 \sinh(\xi) + \eta_2 = \tilde{\eta}_0 + \tilde{\eta}_2 = \kappa\, e^{\tilde{p}_0/\kappa} > 0$.

5.4.3 The Dispersion Relation

Recall from chapter 4 and in particular section 4.3 that we found two different (but equivalent) dispersion relations, differing by a redefinition of mass. One originating from the mass Casimir of the κ-Poincaré algebra and the other from the geodesic distance in de Sitter momentum space. We denoted the two masses

by m and $\tilde{m}$, respectively. Looking at the definition of embedding coordinates, we quickly see that the latter (eq. (4.32)) is given simply by

$$\tilde{m} = \kappa \operatorname{arccosh}\left(\frac{\eta_2}{\kappa}\right), \tag{5.41}$$

from which we can infer, using eq. (4.33), that

$$m^2 = 2\kappa(\eta_2 - \kappa). \tag{5.42}$$

From these representations of the mass, together with the form of the boosts in embedding coordinates, which trivially leave η_2 invariant, it is a trivial consequence that the mass (whichever one we choose) is indeed invariant under boosts. It is also clear that a particle is massless if and only if its η_2 coordinate is equal to κ. It is massive if $\eta_2 > \kappa$. Particles with $\eta_2 < \kappa$ are not physical, since this corresponds to an imaginary mass. This shows again that momentum space is a proper subspace of dS space, and it makes it easy to visualize this subspace in Minkowski embedding coordinates, which we already did in fig. 4.1.

Chapter 6

κ-Poincaré: Multiple Interactions

The backreaction method as outlined in section 5.2.1 ensures that the κ-Poincaré model is invariant under all finite κ-Poincaré transformations as long as each particles undergoes at most one interaction. A problem arises, however, when particles are allowed to undergo multiple interactions. In this chapter we identify this problem and propose a solution for a large class of scenarios, although there will remain situations in which invariance is lost. This is the main result of the research done for this thesis.

6.1 Exposition of the Problem

Consider the situation that a particle with momentum k decays into two particles of momenta p and q, and that the q particle subsequently decays into two particles of momenta r and s. The situation is summarized in the diagram in fig. 6.1, where time flows to the right, as indicated by the arrows (these have nothing to do with spinors). Since the momentum conservation law is noncommutative we also need to specify the 'particle hierarchy' at each interaction vertex. We will always use the convention that the order of incoming (outgoing) particles is from top to bottom in the diagram when one is drawn. Hence the momentum conservation laws at the two vertices corresponding to the diagram in fig. 6.1 read

$$k = p \oplus q, \qquad q = r \oplus s, \tag{6.1}$$

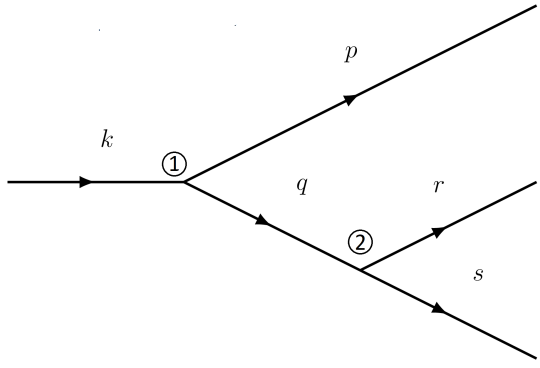


Figure 6.1: Example interaction diagram

respectively. We now focus on the q particle, because it interacts more than once in its lifetime. At the second vertex, since q is the only incoming particle, it obviously is to be treated as the *first* of the incoming particles in this vertex. According to the backreaction method, this means that under a boost of bare rapidity ξ (recall that the bare rapidity is defined to be the rapidity with which the first incoming and first outgoing particles transform), the q particle transforms simply with rapidity ξ. In contrast, at the first vertex the q particle, according to the vertical order in the diagram, is to be treated as the *second* outgoing particle, so it gets a backreaction from the first outgoing particle and hence it should transform not with rapidity ξ but with $\xi \triangleleft p$. The two vertices contradict one another: since $\xi \neq \xi \triangleleft p$, it would look like the two extremes of the worldline of the particle with momentum q transform differently, which would make the transformation rule of the worldline ill-defined.

6.2 And a Solution

For a large class of scenarios, the problem outlined above can be solved by an extension of the backreaction method of section 5.2.1 (which works for individual vertices) to entire interaction diagrams like the one above. Here we will explain this procedure. We will start with a couple of simple examples, which are probably sufficient to get the idea across. In section 6.2.4, a proof is presented that this

procedure works (i.e., is well-defined and renders the conservation law invariant under boosts) for at least all tree-level diagrams, as well as for all completely orderable diagrams, a notion that will be defined later. We will consider only connected diagram, which, from a physics perspective, corresponds to scenarios in which all particles are causally connected to each other. Any general diagram can be pieced together from some collection of connected diagrams, without causing problems in the transformation behavior.

6.2.1 A First Example

We consider again the diagram in fig. 6.1. As illustrated, the backreaction method from section 5.2.1 does not work, so what we will do differently is to allow each of the two vertices 1 and 2 to have a different 'bare' rapidity – let us call them ξ_1 and ξ_2, respectively – instead of requiring that $\xi_1 = \xi_2 = \xi$. Next we simply apply the conventional backreaction method to each vertex, which yields a compatibility equation between the two bare rapidities,

$$\xi_2 = \xi_1 \triangleleft p, \tag{6.2}$$

because q should transform with rapidity ξ_2 (according to vertex 2) as well as with $\xi_1 \triangleleft p$ (according to vertex 1). This compatibility condition replaces the contradiction $\xi = \xi \triangleleft p$ that we would have ended up with, had we used the original backreaction method. Hence if the bare rapidity of vertex 1 is given, the *link* between 1 and 2 provided by the q particle uniquely determines the bare rapidity of vertex 2. And it also works the other way around, since we can rewrite the compatibility equation as

$$\xi_1 = \xi_2 \triangleleft (\ominus p) \tag{6.3}$$

by virtue of the fact that the momenta form a group under the composition law $\oplus$ and the fact that the backreaction $\triangleleft$ is a right action of this group. Since the bare rapidity of a vertex is determined uniquely by any of the rapidities of the particles attached to it, it is enough to specify the rapidity of any of the particles in the diagram to determine the transformation behavior of any of the other particles in the diagram. For the present situation we find that the different momenta transform as follows:

$$\begin{aligned} k \to &\quad \Lambda_{\xi_1}(k), \\ p \to &\quad \Lambda_{\xi_1}(p), \\ q \to &\quad \Lambda_{\xi_1 \triangleleft p}(q), \\ r \to &\quad \Lambda_{\xi_1 \triangleleft p}(r), \\ s \to &\quad \Lambda_{(\xi_1 \triangleleft p) \triangleleft r}(s) = \Lambda_{\xi_1 \triangleleft (p \oplus r)}(s). \end{aligned}$$

Then all boosted momenta are well-defined and the momentum composition law behaves covariantly under boosts. This method, let us call it the 'pairwise method', of assigning a rapidity to each particle in the diagram is not the only way to do this, in general. Note that no lines in the present diagram cross each other (even if we imagine them infinitely extended to the right, for outgoing particles, or to the left, for incoming particles). When that is the case, we may in fact interpret the above procedure of determining the rapidities equivalently in an alternative way, which we will call the 'total momentum method':

Alternatively, instead of assigning a bare rapidity to each vertex, ξ_1, ξ_2, we now assign only a single bare rapidity ξ to the whole diagram, namely the rapidity with which the topmost incoming particle transforms[1]*, and that all particles in the entire diagram 'backreact' on each other: any given incoming (outgoing) particle gets a backreaction from the total momentum constituted by all particles above the considered particle in the diagram, summed in the correct order.*

The key thing to note here is that instead of looking at the set of momenta at each vertex separately, we include in the computation at each vertex the complete set of momenta of the whole diagram. This is very similar to what happens with translations if one includes multiple interactions. In that case one has to use, at each vertex, the composition law governing the *total* momentum in the entire diagram (see section 2.2) instead of of the composition law of 'just' that vertex. This parallel between the implementation of translations and boosts suggests that the universe might be a lot more interconnected than one is usually inclined to think. In any case, this is true for the κ-Poincaré model.
The second formulation, the total momentum method, works only when particle

[1] Assuming the diagram contains a finite number of particles, and imagining all particle lines infinitely extended to the right, for outgoing particles, or to the left, for incoming particles, there always is a topmost incoming particle, because the lines do not cross, by assumption.

lines in the diagram do not cross, even if we imagine them infinitely extended to the right, for outgoing particles, or to the left, for incoming particles. So the initial procedure is more general. The second formulation is mainly interesting because it shows very quickly how each particle should transform, and because it draws a parallel with the translations as just alluded to. In the particular case of the present diagram it can quickly be verified that the two methods yield the same result. A more interesting example is provided next.

6.2.2 A Loop Example

As another example we consider the loop diagram diagram in fig. 6.2. Again the ordering of the particles in each vertex is by convention from top to bottom in the diagram. Since there are no crossing lines in this case we can use the 'total momentum' method, in addition to the generic 'pairwise' method, to find the rapidities with which each of the particles transforms under a boost. So we will analyze the diagram twice, using each of the methods, both yielding the same results.

The Pairwise Method

Let the bare rapidity if vertex i be ξ_i for each $i = 1, 2, 3, 4$. With $V = 4$ vertices and $L = 1$ loop we have $(V-1)+L = 4$ compatibility conditions for the rapidities. These are found by imposing that the particles that connect the vertices have a well-defined momentum transformation, and read

$$\xi_2 = \xi_1 \triangleleft p, \qquad \xi_3 = \xi_1, \qquad \xi_4 = \xi_3 \triangleleft \ell, \qquad \xi_2 = \xi_4 \triangleleft \ell'. \tag{6.4}$$

Note that because of the loop, we have two different compatibility equations involving ξ_2! This is a general consequence of loops. So let us first make sure that the two equations are consistent in this case. Substituting the third equation into the fourth, the latter equation becomes $\xi_2 = (\xi_3 \triangleleft \ell) \triangleleft \ell' = \xi_3 \triangleleft (\ell \oplus \ell')$, and by conservation of momentum at vertex 3, we see that this equation is equivalent to the first compatibility equation. Therefore the momentum transformation is well-defined for each particle. Summarizing, the different particles transform

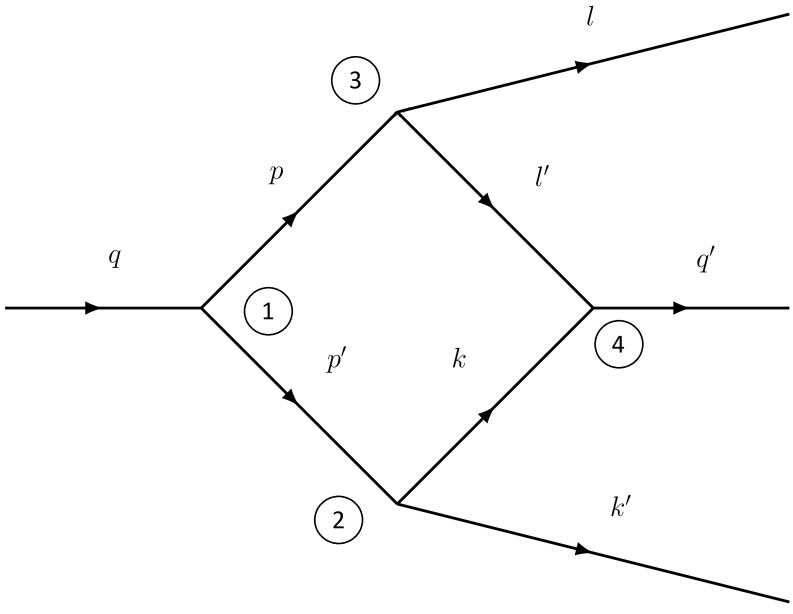


Figure 6.2: Example loop diagram

with the following rapidities, in terms of ξ_1.

$$q \to \Lambda_{\xi_1}(q), \tag{6.5}$$
$$p \to \Lambda_{\xi_1}(p), \tag{6.6}$$
$$p' \to \Lambda_{\xi_1 \triangleleft p}(p') \tag{6.7}$$
$$\ell \to \Lambda_{\xi_1}(\ell), \tag{6.8}$$
$$\ell' \to \Lambda_{\xi_1 \triangleleft \ell}(\ell'), \tag{6.9}$$
$$k \to \Lambda_{\xi_1 \triangleleft p}(k), \tag{6.10}$$
$$k' \to \Lambda_{\xi_1 \triangleleft p \triangleleft k}(k') \tag{6.11}$$
$$q' \to \Lambda_{\xi_1 \triangleleft \ell}(q'). \tag{6.12}$$

The Total Momentum Method

In this method we work with only a single bare rapidity, ξ and each particle gets a backreaction from all particles above it in the diagram. For definiteness we will suppose that vertex 2 happens 'before' vertex 3[2]. That means that we get the following transformations:

$$q \to \Lambda_{\xi}(q), \tag{6.13}$$
$$p \to \Lambda_{\xi}(p), \tag{6.14}$$
$$p' \to \Lambda_{\xi \triangleleft p}(p'), \tag{6.15}$$
$$\ell \to \Lambda_{\xi}(\ell), \tag{6.16}$$
$$\ell' \to \Lambda_{\xi \triangleleft \ell}(\ell'), \tag{6.17}$$
$$k \to \Lambda_{\xi \triangleleft p}(k) \quad \text{(according to vertex 2) and} \tag{6.18}$$
$$k \to \Lambda_{\xi \triangleleft (l \oplus l')}(k) \quad \text{(according to vertex 4)}, \tag{6.19}$$
$$k' \to \Lambda_{\xi \triangleleft (\ell \oplus \ell' \oplus k)}(k') = \Lambda_{\xi \triangleleft (p \oplus k)}(k') = \Lambda_{\xi \triangleleft p \triangleleft k}(k'), \tag{6.20}$$
$$q' \to\to \Lambda_{\xi_1 \triangleleft \ell}(q'). \tag{6.21}$$

We have two formulations of the transformation of k, because while this particle travels from vertex 2 to vertex 4, vertex 3 'happens in between'. Therefore,

[2]To do the computation one has to pick an order in time of the different vertices. This is in the end irrelevant, and all orders compatible with incoming and outgoing particles yield the same result.

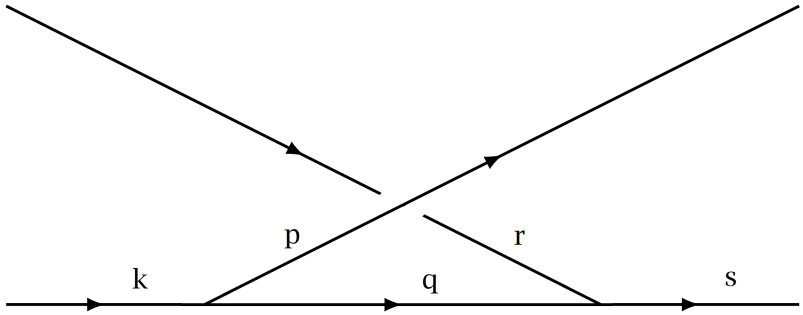


Figure 6.3: Example interaction diagram with crossing lines, where the pairwise method of assigning boost rapidities is applicable, but the total momentum method is not. Physically diagrams like this represent interaction scenarios where global momentum conservation is violated.

according to vertex 2, k should transform as if vertex 3 had not occurred yet, while according to vertex 4 k should transform as if vertex 3 had already happened. But these two formulations are of course equivalent, by conservation of momentum at vertex 3. Thus each rapidity is well-defined. We also see that the result is equivalent to the transformation we found in the previous method if we identify $\xi = \xi_1$. This example shows how loops can potentially lead to problems, which is most evident from the pairwise method, because the result of the loop is that there are *two* compatibility conditions for ξ_2, instead of just one. These two conditions might not have been consistent with each other. Nevertheless we have seen that in this particular case they are consistent, so even though loops may in general lead to problems, there are also loop diagrams that can be handled perfectly well by our proposed methods.

6.2.3 An Example with Crossing Lines

The last example diagram, fig. 6.3, is one which cannot be drawn without crossing lines. At least not in a way that respects the particle hierarchy at each vertex and our conventions: time flows to the right and the order of particles is from top to bottom. One way one might be tempted to draw the diagram without

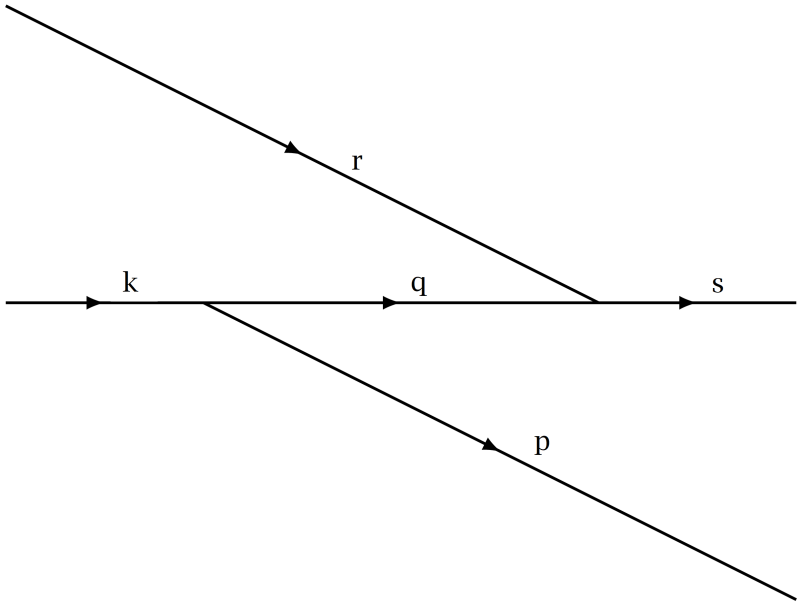


Figure 6.4: An interaction diagram that one might obtain by trying to represent the diagram in fig. 6.3 without crossing lines. The resulting diagram, however, is *physically different* from the one in fig. 6.3, because of the noncommutativity of the κ-Poincaré momentum conservation law.

crossing lines is to draw the p line that emerges from the left vertex *below*, instead of above, the q line, as illustrated in fig. 6.4. Then there are indeed no crossing lines anymore, but the order of the two outgoing particles in the left vertex has now been changed, which matters because the momentum composition law is noncommutative. Hence the diagram obtained in this way is really a different diagram than the original one.

The fact that the diagram in fig. 6.3 cannot be drawn without crossing lines implies that the total momentum method is not applicable (or at least that we cannot guarantee that it is). We will illustrate below why this is the case. On the other hand, the diagram is of tree-level, so the pairwise method will work perfectly fine. We will also work through this method below.

The Pairwise Method

Employing the pairwise method, we start by assigning to each vertex its own bare rapidity, the left vertex ξ_1 and the right vertex ξ_2. The compatibility condition is then provided by the q particle, since it connects the two vertices, and it reads $\xi_1 \triangleleft p = \xi_2 \triangleleft r$, so that we can write

$$\xi_2 = \xi_1 \triangleleft (p \ominus r). \tag{6.22}$$

We therefore find that the particles transform as follows in terms of the bare rapidity ξ_1,

$$k \to \Lambda_{\xi_1}(k), \tag{6.23}$$
$$p \to \Lambda_{\xi_1}(p), \tag{6.24}$$
$$q \to \Lambda_{\xi_1 \triangleleft p}(q), \tag{6.25}$$
$$r \to \Lambda_{\xi_1 \triangleleft (p \ominus r)}(r), \tag{6.26}$$
$$s \to \Lambda_{\xi_1 \triangleleft (p \ominus r)}(s). \tag{6.27}$$

The Total Momentum Method

Now let us check that the total momentum method does not work in this case, as a result of the crossing lines. Recall that in this method there is a single bare rapidity ξ and each particle gets a backreaction from the total momentum constituted by all the particles *above* the considered particle in the diagram. This means that k and p get a backreaction from r, and that q gets a backreaction from $r \oplus p$. Identifying $\xi_1 = \xi \triangleleft r$ this leads to the same transformations for k, p and q as we obtained from the previous method. But let us now consider the second vertex. Here the p particle is above the r particle, so r gets a backreaction from p, s also gets a backreaction from p, and q gets a backreaction from $p \oplus r$. That means in particular that q transforms with rapidity $\xi \triangleleft (p \oplus r)$ according to the second vertex, whereas according to the first vertex it transformed with rapidity $\xi \triangleleft (r \oplus p)$. The noncommutativity of the κ-Poincaré composition law therefore leads to the conclusion that the total momentum method does not yield a well-defined rapidity for the q particle and hence the method cannot be used. This is a general feature of diagrams with crossing lines. Note that for the same

reason the transformation behavior of the q particle under *translations* (see section 2.2) is ill-defined as well, so if one were to require translation invariance then diagrams like this would be forbidden anyway and there would be no problems with the definition of boosts at all.
Another general feature of diagrams with crossing lines is that they often violate global momentum conservation. We already encountered this in section 2.2 as well when discussing the implementation of translations, and here we encounter it again. The total incoming momentum is $r \oplus k$, while the total outgoing momentum is

$$p \oplus s = (k \ominus q) \oplus s = k \ominus q \oplus r \oplus q \neq r \oplus k. \tag{6.28}$$

The fact that diagrams with crossing lines do not behave covariantly under boosts and translations and that they violate global momentum conservation arguably suggest that these crossing-line diagrams might not be allowed in nature.

6.2.4 The General Case

In this section we provide the proof for the general case: for any *tree-level* diagram and for any *ordered* diagram there is a way to assign a rapidity to each particle in the diagram in a well-defined way, such that if each particle's momentum transforms according to the corresponding rapidity, and the original momenta satisfy the conservation law in each vertex, then the boosted momenta satisfy the conservation law in each vertex as well, meaning that the theory is invariant under boosts. By *tree-level* we mean that the diagram contains no loops, i.e., no pair of vertices is connected by more than one path. And by an *ordered* diagram we mean a diagram where lines do not cross, even if we imagine the relevant lines infinitely extended to either the left (for incoming particles) or the right (for outgoing particles). With the latter type of diagram there is always an overall order that can be assigned to the particles in the diagram which is also consistent with the ordering in each individual vertex, hence the name. We will focus here only on *connected* diagrams (i.e., diagrams where each pair of vertices is connected by at least one path). For any diagram which is not connected can be pieced together, without the problem of conflicting compatibility equations, from multiple connected diagrams. From a physics point of view this makes sense as well, as the different connected parts of a general diagram cannot

causally influence each other.

Beginning our proof, we consider the most general form of a tree level diagram, which can be described in the following way. Start with an arbitrary vertex; it has a finite number of incoming particles with momenta $p^1, p^2, \ldots$, and outgoing particles with momenta $p_1, p_2, \ldots$. (Here the indices are placed up or down to show that a particle is incoming or outgoing.) We assume without loss of generality that the particles are numbered such that p_i appears in the conservation law as the i^{th} outgoing particle, and similarly for the incoming ones. We assign a rapidity ξ to this vertex, by which we mean, by definition, that p_1 and p^1 transform with rapidity ξ. Invariance of the conservation law then determines how the other particles in the ξ-vertex must transform. Namely, p_i and p^i need to transform with rapidities $\xi \triangleleft (p_1 \oplus p_2 \oplus \cdots \oplus p_{i-1})$ and $\xi \triangleleft (p^1 \oplus p^2 \oplus \cdots \oplus p^{i-1})$, respectively. Now since we are dealing with a generic tree-level diagram, each p_i that is created in the ξ-vertex might or might not be annihilated again in another vertex. If this is the case, we assign a bare rapidity ξ_i to this other vertex, and we refer to it as the ξ_i-vertex. Of course this vertex again has a finite number of incoming and outgoing particles, the momenta of which we denote by ${p_i}^1, {p_i}^2, \ldots$ and $p_{i1}, p_{i2}, \ldots$, respectively. Repeating this procedure, now starting from each ξ_i vertex, and so on, we get a description of the whole (connected) diagram. The particle with momentum ${p_i}^{jk}{}_{\ell}$, for instance, is by definition the particle at which we arrive by starting in the ξ-vertex, following the i^{th} outgoing particle until we are at the ξ_i vertex, then following there the j^{th} incoming particle, (opposite its propagation direction) until we arrive at the ${\xi_i}^j$-vertex, then following there the k^{th} incoming particle, (opposite its propagation direction) until we arrive at the ${\xi_i}^{jk}$-vertex, and then finally picking the ℓ^{th} outgoing particle in that vertex. This procedure yields a unique description of each particle, by virtue of the fact that, by assumption, there are no loops present.

Our task is to make sure that each particle in the diagram has a *well-defined* boost parameter associated to it, determined by the compatibility conditions. We consider first the particle p_i, which is the i^{th} outgoing particle in the ξ-vertex. This particle (assuming that it does not fly off to infinity, in which case there is nothing to check) is also an incoming particle in the ξ_i vertex, say the

jth one. Thus, according to the one vertex, p_i should transform with rapidity

$$\xi \triangleleft (p_1 \oplus p_2 \oplus \cdots \oplus p_{i-1}) \,, \tag{6.29}$$

whereas according to the other vertex it should transform with rapidity

$$\xi_i \triangleleft \left(p_i{}^1 \oplus p_i{}^2 \oplus \cdots \oplus p_i{}^{j-1}\right) . \tag{6.30}$$

For the rapidity to be well-defined, these two expressions should coincide. Using the fact that $\oplus$ makes the momentum manifold into a group, and that $\triangleleft$ is a right action of this group, we can formulate this requirement by saying that

$$\begin{aligned} \xi_i &= \xi \triangleleft \left(p^1 \oplus p^2 \oplus \cdots \oplus p^{i-1}\right) \triangleleft \ominus \left(p^i{}_1 \oplus p^i{}_2 \oplus \cdots \oplus p^i{}_{j-1}\right) && (6.31) \\ &= \xi \triangleleft \left(p^1 \oplus p^2 \oplus \cdots \oplus p^{i-1} \ominus \left(p^i{}_1 \oplus p^i{}_2 \oplus \cdots \oplus p^i{}_{j-1}\right)\right) . && (6.32) \end{aligned}$$

Thus, once ξ is specified, all the ξ^i are determined in terms of ξ by means of this formula, and this ensures that the transformation of all the p^i particles is well-defined. A similar argument applies of course to the incoming particles p_i of the ξ-vertex. Hence in this way we can ensure that all particles *directly attached* to the ξ-vertex have well-defined transformation behavior.

Having ensured this, we go on to one of the other vertices, one that is directly linked to ξ. Say we look at the ξ_i-vertex. Repeating the exact same argument as the one used for the ξ-vertex, we find that the value of ξ_i determines the rapidities of all vertices directly linked to *it* (that is, $\xi_i{}^j$ and ξ_{ik} for all j, k), by similar formulas as the ones above for ξ. Notice, however, that, since ξ is directly linked to ξ_i, the value of ξ_i also imposes a certain value of ξ. But it is easily checked that the way it does this is consistent with how ξ determined ξ_i in the first place, so there is no problem here. This then ensures that all particles connected to the ξ_i-vertex have well-defined transformation behavior under boosts.

It is clear that we can continue in this way, going through all vertices, first all direct links of ξ, then all direct links of direct links of ξ_i, and so on, each time determining the values of the directly linked rapidities in terms of the rapidity of the considered vertex, and hence ultimately in terms of the original ξ. And we may do this without contradicting the rapidities that we had already defined

in previous steps, because each vertex will only be connected to ξ by a single possible path (because the diagram is of tree-level, by assumption). Thus we conclude that by defining the rapidities in this way for the entire diagram, in terms of ξ, we have ensured that the whole diagram transforms in a well-defined way under boosts, such that the momentum conservation law at each vertex is invariant under the boosts. So let us conclude:

With the above implementation of boosts on tree-level diagrams, and with translations acting in the original way, the κ-Poincaré model, restricted to tree-level interaction diagrams, is invariant under all finite κ-Poincaré transformations.

Next, the proof for ordered diagrams is simple. We may use the total momentum method in this case and all we need to check is that a particle that interacts twice has a well-defined boost rapidity associated to it. Because the diagram has no crossing lines, there is an overall particle ordering (namely the order as drawn from top to bottom in the diagram) which is consistent with the order in each individual vertex. Conservation of the total momentum constituted by all particles above a given particles in the diagram guarantees that when a particle is connected to two vertices then the rapidity induced by the first vertex is the same as the rapidity induced by the second vertex. Hence the total momentum method always works in this case as well.

Some Technical Remarks

Note that in the tree-level scenario we do not need to assume that the diagram is finite. If we know the rapidity with which one of the particles transforms, then we can infer the rapidity needed for all other particles, even if there are infinitely many. We do need to assume, however, that the number of vertices is countable, as we have used an induction argument, but from a physics perspective that is always the case anyway. Also, we have to assume that each vertex has only finitely many direct links, i.e., only a finite number of particles can partake in each interaction. But to doubt this would be a bit far fetched as well.
Note also that we could have started the construction of the proof with any vertex in the diagram, and the rapidity of an any given particle in that vertex. Given that one rapidity, one is able to determine the bare rapidity of that vertex, and

consequently the rapidity of each particle in the diagram. So if one knows for any given single particle how it transforms then that is enough to determine the transformation behavior of all other particles in the diagram. Therefore one can define a boosted observer, for instance, by saying that that some given particle transforms with a given rapidity. Then the rapidities of all other particles in the diagram are determined uniquely and can be computed explicitly, provided the momenta are known[3].
When using the total momentum method however, we do need to require that the number of particles present in the diagram at any time is finite, because otherwise we cannot compute the total momentum, as the total energy would be infinite (because of the triviality of the zeroth component of the addition law of momenta).

6.2.5 The Difficulties with Loops

An essential ingredient that we have used in the proof for tree-level diagrams is that for those diagrams each vertex can be traced back to the original vertex ξ via one and only one 'path'. This is the reason that all rapidities are well-defined. If we allow for loops, however, there will be multiple paths between some pairs of vertices, and these different paths will in general lead different compatibility conditions for these pairs of vertices, leading potentially to ill-defined rapidities. This is the problem we face with loops. For a general loop diagram the method cannot be guaranteed to work. Nevertheless, in some loop diagrams there is no problem at all, which is the case, for instance, when the total momentum method is applicable, as we already saw in the example in section 6.2.2.
Moreover, a lot of loop diagrams for which the described method *a priori* does not seem to work, can be modified by changing the order of the particles in a vertex in such a way that the method *does* work. Note that this changes the momentum conservation laws, by nonassociativity, but nature might well have decided that of all the possible particle orderings in the vertices only those that behave covariantly are allowed. This possibility is reinforced by the observation

[3]Of course, one needs to know the vertex structure in order to do this. Conversely, if one does not know the vertex structure, one might compare results between two relatively boosted observers to experimentally determine what the vertex structure must have been in a given process.

that a lot of the non-covariant diagrams also violate global momentum conservation. The most viable explanation for the presence of non-covariant diagrams might therefore be that they are simply excluded from nature. Assuming this is indeed the case, the κ-Poincaré model becomes a completely relativistic theory.

Chapter 7

An Alternative Conservation Law

In the previous chapter we extended the κ-Poincaré model to a κ-Poincaré invariant theory that can deal with (a certain class of) multi-vertex diagrams. Already in chapter 5, however, we found that boosts were not compatible with the Relative Locality Framework (RLF) when interactions are taken into account. The reason for this is in essence the backreaction, which originates from the momentum composition law. But interestingly, there is an alternative candidate for the composition law [28], which is κ-Poincaré invariant *without requiring a backreaction*, so with this alternative composition law all particles conveniently transform with the same boost rapidity[1]. In this chapter we discuss the model that arises if we abandon the original κ-Poincaré composition law in favor of this alternative one, while leaving the rest of the model unchanged. Although this ensures that the boosts are RLF symmetries, translation are not, because momentum space is now not a group under the new composition law, a fact which we often used in proving the well-definition of the RLF translations in section 2.2.

7.1 The Conservation Law

In [28] the authors derive a κ-Poincaré invariant composition law on de Sitter (momentum) space. They use Minkowski embedding coordinates, in which the the boosts have a much simpler form, namely just that of a Minkowski boost, as

[1] That is, requiring that the alternative composition law indeed behaves covariantly.

we discussed in section 5.4. Finding a κ-Poincaré invariant composition law then amounts to finding a composition law that is invariant under this Minkowski boost, but with one extra requirement: if we are working on Minkowski space we need to make sure that the composition of two momenta is again a momentum, that is, that the composition of two points in the subspace of physical momenta is again in this subspace. This subspace consists of those points (η_0, η_1, η_2) in Minkowski space that satisfy $\eta_0 + \eta_2 \geq \kappa$ (this is where the comoving dS coordinates are defined and energies are nonnegative) and $\eta_2 > 0$ (this corresponds to particles of real mass), see figure 7.1. And, lastly, in our case it is important to check if the composition law has the correct $\kappa \to \infty$ limit in comoving coordinates, but recall that this limit does not make much sense in embedding coordinates, so we first have to transform back to comoving coordinates before we take this limit.

In 1+1 dimension dS space, the composition law derived in [28] reads, in embedding coordinates,

$$(\eta \boxplus \zeta)_0 = \eta_0 + \zeta_0 - \eta_0 \frac{\zeta_1(\eta_1(\kappa+\zeta_2)+\zeta_1(\kappa+\eta_2)) - \zeta_0(\eta_0(\kappa+\zeta_2)+\zeta_0(\kappa+\eta_2))}{\kappa(\kappa+\eta_2)(\kappa+\zeta_2)} \tag{7.1}$$

$$(\eta \boxplus \zeta)_1 = \eta_1 + \zeta_1 - \eta_1 \frac{\zeta_1(\eta_1(\kappa+\zeta_2)+\zeta_1(\kappa+\eta_2)) - \zeta_0(\eta_0(\kappa+\zeta_2)+\zeta_0(\kappa+\eta_2))}{\kappa(\kappa+\eta_2)(\kappa+\zeta_2)} \tag{7.2}$$

$$(\eta \boxplus \zeta)_2 = \frac{2\eta_2\zeta_2 - \eta_\mu\eta^\mu}{\kappa} = \frac{\eta_0\zeta_0 - \eta_1\zeta_1 + \eta_2\zeta_2}{\kappa}. \tag{7.3}$$

We have used the symbol $\boxplus$ to distinguish it from the κ-Poincaré composition law $\oplus$.

We state here without proof that the κ-Poincaré momentum space is indeed closed under the composition law $\boxplus$, just as it was under $\oplus$. Explicitly, this means that for all pairs of triplets $(\eta_0, \eta_1, \eta_2) \in \mathbb{R}^3$ satisfying the equations $-\eta_0^2 + \eta_1^2 + \eta_2^2 = \kappa^2$, $\eta_2 \geq \kappa$ and $\eta_0 + \eta_2 \geq \kappa$, the $\boxplus$ sum of the pair also satisfies these equations. A similar statement was provided in [28], but it was not discussed there that real masses are also preserved by the conservation law.

Since the composition law (7.1),(7.2),(7.3) is well-defined as a composition law on our momentum space, we may express it in comoving coordinates. It then

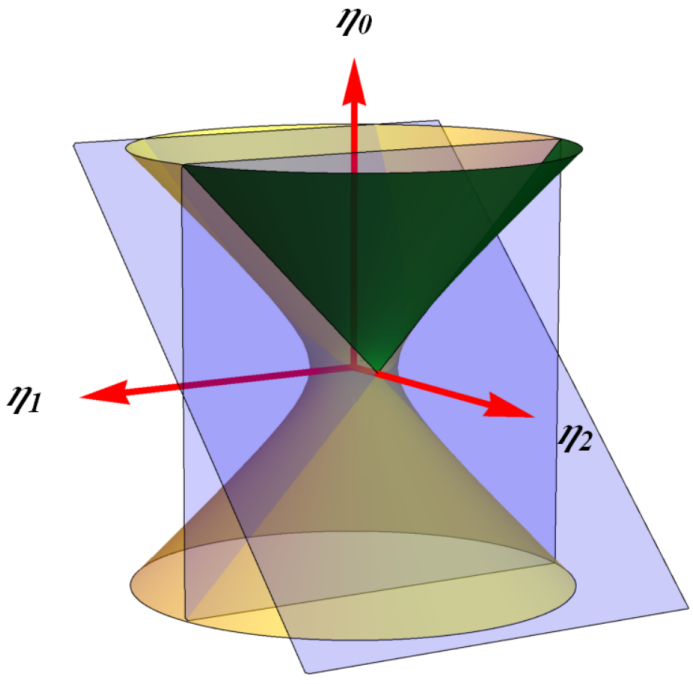


Figure 7.1: κ-Poincaré momentum space in 1+1 dimensions, embedded in 2+1 dimensional Minkowski space. The hyperboloid consisting of the green and yellow patches represents the entire dS space. The part of dS space above the diagonal blue ($\eta_0 + \eta_2 = \kappa$) plane is the part of dS space which is described by comoving coordinates (p_0, p_1). And the part of this which, additionally, lies in front of the vertical blue ($\eta_2 = 0$) plane corresponds to momenta with a *real* mass; this part, the actual κ-Poincaré momentum space, is colored green.

has the form

$$(p \boxplus q)_0 = \kappa \log \left(\frac{e^{\frac{p_0+q_0}{\kappa}} \left(\kappa^2 + e^{p_0/\kappa}\left(\kappa^2 - p_1 q_1\right)\right)^2 - \kappa^2 p_1^2 e^{\frac{3p_0-q_0}{\kappa}}}{\kappa^4 + 2\kappa^4 e^{p_0/\kappa} + \kappa^2 e^{\frac{2p_0}{\kappa}}(\kappa - p_1)(\kappa + p_1)} \right), \tag{7.4}$$

$$(p \boxplus q)_1 = \frac{\kappa^2 p_1 e^{p_0/\kappa}\left(\kappa^2 + \kappa^2 e^{p_0/\kappa} + p_1^2\left(-e^{\frac{2p_0}{\kappa}}\right)\right)}{e^{\frac{p_0+2q_0}{\kappa}}\left(\kappa^2 + e^{p_0/\kappa}\left(\kappa^2 - p_1 q_1\right)\right)^2 - \kappa^2 p_1^2 e^{\frac{3p_0}{\kappa}}} \tag{7.5}$$

$$+ \frac{e^{\frac{2q_0}{\kappa}}\left(p_1 e^{\frac{2p_0}{\kappa}}\left(\kappa^2 - p_1 q_1\right) + \kappa^2 q_1 e^{p_0/\kappa} + \kappa^2 q_1\right)\left(\kappa^2 + e^{p_0/\kappa}\left(\kappa^2 - p_1 q_1\right)\right)}{e^{\frac{p_0+2q_0}{\kappa}}\left(\kappa^2 + e^{p_0/\kappa}\left(\kappa^2 - p_1 q_1\right)\right)^2 - \kappa^2 p_1^2 e^{\frac{3p_0}{\kappa}}}, \tag{7.6}$$

and expresses a deformed momentum conservation law, which satisfies the crucial property that it is invariant under the κ-Poincaré boosts (4.21), (4.22) in the ordinary sense, i.e., the implication

$$p \boxplus q = k \qquad \Rightarrow \qquad \Lambda_\xi(p) \boxplus \Lambda_\xi(q) = \Lambda_\xi(k) \tag{7.7}$$

holds for any rapidity ξ and there is no need for a backreaction, in contrast to the κ-Poincaré conservation law $\oplus$. In this form it can also be checked that the Special Relativistic limit is the correct one. Interestingly the symmetrized (commutative) version of the addition law, which reads

$$(p \boxplus q)_0 = p_0 + q_0 - \frac{p_1 q_1}{\kappa} + \frac{1}{4\kappa^2}(\mathrm{p1} - \mathrm{q1})(\mathrm{p1q0} - \mathrm{p0q1}), \tag{7.8}$$

$$(p \boxplus q)_1 = p_1 + q_1 - \frac{p_0 q_1 + p_1 q_0}{\kappa} + \frac{1}{4\kappa^2}\left(3\mathrm{p0}^2\mathrm{q1} + \mathrm{p0q0}(\mathrm{p1} + \mathrm{q1})\right) \tag{7.9}$$

$$+ \frac{1}{4\kappa^2}\left(\mathrm{p1}\left(2\mathrm{q1}(\mathrm{p1} + \mathrm{q1}) + 3\mathrm{q0}^2\right)\right), \tag{7.10}$$

to second order in $1/\kappa$, coincides to first order with the commutative composition laws found in [23] (equations 78 and 84), but not to second order.

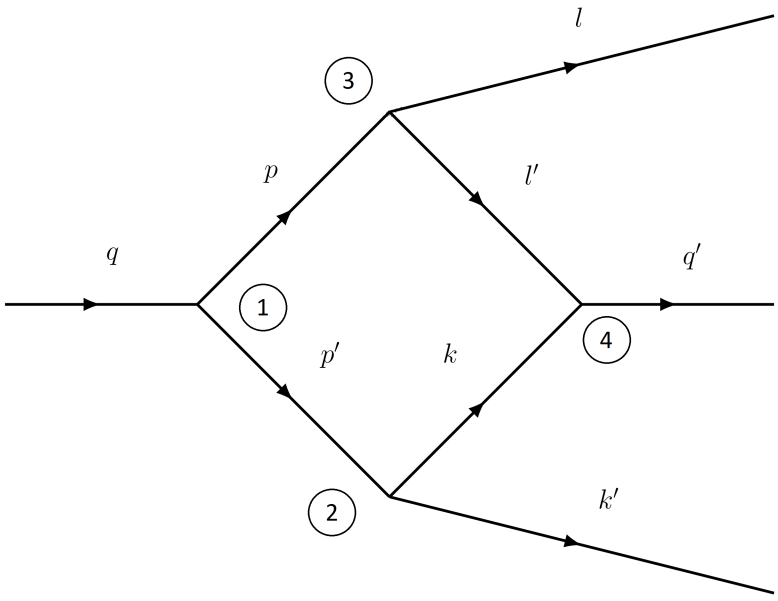
7.1.1 Nonassociativity of the Composition Law

The fact that the composition law is nonassociative leads to an ambiguity in the model, in addition to the ambiguity due to noncommutativity that by now we

have gotten used to. One possibility is to always compose momenta from left to right, i.e., start with the the left most momentum, add to it the momentum to the right of it, then add to that the momentum to the right of the first two, one, and so on. This says that the 'nonassociativity order' is the same as the 'noncommutativity order'. But instead it might also be the case that the nonassociativity order is completely independent from the noncommutativity order. We will not comment on this issue any further.

7.2 Translations

The translations that were introduced in section 2.2 using the total momentum at each vertex for the purpose of being RLF symmetries, do not work as well together with the momentum conservation law of this chapter. The reason is basically that, in contrast to $\oplus$, $\boxplus$ does not make momentum space into a group. For instance in the diagram below, the conservation law at vertex 4 is $\ell' \boxplus k = q'$. Because of the lack of inverse elements, we cannot guarantee that this is equivalent to the 'total momentum expression' $(l \boxplus (\ell' \boxplus k)) \boxplus k' = (l \boxplus q') \boxplus k'$. And even if it was, then, because of nonassociativity, it would be not be equivalent to, for instance, $((l \boxplus \ell') \boxplus k) \boxplus k' = (l \boxplus q') \boxplus k'$, in which a different nonassociativity order is used in the composition. Clearly this will become a problem in a lot of diagrams, and it means that we cannot formulate all vertex conservation laws in terms of a total momentum. This, in turn, means that we cannot define translations in a well-defined way such that they are symmetry of the Relative Locality Framework. Hence the best option at this point is to just use our good old original κ-Poincaré translations $x^\mu \to x^\mu + a^\mu$. Although those are not full RLF symmetries either, because they do not leave the RLF interaction equation invariant, they are well-defined symmetries of the non-interacting theory as well as of the momentum space description of interactions (that is, the momentum conservation law).



7.3 κ-Poincaré Boosts Are RLF Symmetries

Since using the total momentum at each vertex in the conservation law does not work in favor of the translations anymore, it is more natural to choose the original formulation $\mathscr{K} = \left(\boxplus_i p_i\right) - \left(\boxplus_j q_j\right) = 0$ of the conservation law, where the p_i and q_j are simply the incoming and outgoing particles, respectively, at the given vertex. This form of the conservation law is compatible with the boosts (in the sense that boosts are now a symmetry of the whole RLF). Let us prove this. We denote a boost of a particle with momentum p by

$$p \to \tilde{p}, \qquad x^\mu \to x^\nu \frac{\partial p_\nu}{\partial \tilde{p}_\mu} \tag{7.11}$$

In order to make this a symmetry of the whole relative locality framework, we showed in section 5.3.2 that z^μ needs to transform as

$$z \to \tilde{z} = z \frac{\partial \mathscr{K}}{\partial \tilde{p}} \frac{\partial \tilde{p}}{\partial p} \left(\frac{\partial \tilde{\mathscr{K}}}{\partial p} \right)^{-1} = z \frac{\partial \mathscr{K}}{\partial \tilde{p}} \left(\frac{\partial \tilde{\mathscr{K}}}{\partial \tilde{p}} \right)^{-1}, \tag{7.12}$$

written in matrix notation[2]. The second equality is due to the fact that there is no backreaction; the boosted composition law depends only on the original momenta p through the boosted momentum $\tilde{p}$ and not the other boosted momenta. This is the crucial difference with the original κ-Poincaré model, where the backreaction invalidates this statement.

Note that for a boost to be a well-defined transformation of the triple (z, x, p), the conservation law $\mathscr{K}$ must satisfy the requirement that (7.12) is independent of the chosen momentum p of the vertex. To see that this is indeed true, write $\mathscr{K}_p = \boxplus_i p_i$ and $\mathscr{K}_q = \boxplus_j q_j$, so that $\mathscr{K} = \mathscr{K}_p - \mathscr{K}_q$. Now, as above, we have $\tilde{\mathscr{K}}_p = \Lambda_\xi(\mathscr{K}_p)$ and $\tilde{\mathscr{K}}_q = \Lambda_\xi(\mathscr{K}_q)$, and for any p_i we can write

$$\frac{\partial \mathscr{K}}{\partial \tilde{p}_i} \left(\frac{\partial \tilde{\mathscr{K}}}{\partial \tilde{p}_i} \right)^{-1} = \frac{\partial \mathscr{K}_p}{\partial \tilde{p}_i} \left(\frac{\partial \tilde{\mathscr{K}}_p}{\partial \tilde{p}_i} \right)^{-1} = \frac{\partial \mathscr{K}_p}{\partial \tilde{\mathscr{K}}_p} \frac{\partial \tilde{\mathscr{K}}_p}{\partial \tilde{p}_i} \left(\frac{\partial \tilde{\mathscr{K}}_p}{\partial \tilde{p}_i} \right)^{-1} = \frac{\partial \mathscr{K}_p}{\partial \tilde{\mathscr{K}}_p} = \left. \frac{\partial p}{\partial \tilde{p}} \right|_{\tilde{p}=\tilde{\mathscr{K}}_p}, \tag{7.14}$$

which indeed is the same for all p_i. Similarly, for any q_i we can write

$$\frac{\partial \mathscr{K}}{\partial \tilde{q}_i} \left(\frac{\partial \tilde{\mathscr{K}}}{\partial \tilde{q}_i} \right)^{-1} = -\frac{\partial \mathscr{K}_q}{\partial \tilde{q}_i} \left(-\frac{\partial \tilde{\mathscr{K}}_q}{\partial \tilde{q}_i} \right)^{-1} = \frac{\partial \mathscr{K}_q}{\partial \tilde{\mathscr{K}}_q} \frac{\partial \tilde{\mathscr{K}}_q}{\partial \tilde{q}_i} \left(\frac{\partial \tilde{\mathscr{K}}_q}{\partial \tilde{q}_i} \right)^{-1} = \frac{\partial \mathscr{K}_q}{\partial \tilde{\mathscr{K}}_q} = \left. \frac{\partial p}{\partial \tilde{p}} \right|_{\tilde{p}=\tilde{\mathscr{K}}_q}, \tag{7.15}$$

which does not depend on the chosen q_i either. And the two expressions above are also equal to each other by virtue of the fact that conservation of momentum

[2]With the convention that the second index of the Jacobian matrix labels the components of the object *below* the denominator, i.e.,

$$\left(\frac{\partial \mathscr{K}}{\partial p} \right)_\mu^{\ \nu} = \frac{\partial \mathscr{K}_\mu}{\partial p_\nu} \tag{7.13}$$

is invariant under boosts, $\tilde{\mathscr{K}} = \tilde{\mathscr{K}_p} - \tilde{\mathscr{K}_q} = 0$.

Hence with these definitions, the κ-Poincaré boosts are well-defined symmetry of the whole Relative Locality framework.

Chapter 8

Discussion and Outlook

In this thesis we have described how the κ-Poincaré model can be extended in such a way that it can deal with particles that undergo multiple interactions during their lifetimes. With our proposed generalized implementation of boosts, a large class of interaction diagrams with multiple vertices now behaves covariantly under the κ-Poincaré transformation group, at least when translations are also implemented in the correct way. Still not all interaction diagrams behave covariantly, however, and it is still an unanswered question how to interpret this. One possibility is that nature only allows processes that do in fact behave covariantly; another is that the κ-Poincaré transformations are not symmetries of nature after all. And, of course, there is also the possibility that the κ-Poincaré model is simply not realized in nature. Future research in quantum gravity might shed light on this question. We hope that the κ-Poincaré model can someday be derived as the $\hbar \to 0, G \to 0$ limit of some fundamental theory of quantum gravity.

Also we have compared the κ-Poincaré model to the de Sitter momentum space model in the Relative Locality framework (RLF). We have seen that for free particles the two models coincide, up to an irrelevant redefinition of mass. For interacting particles, however, the RLF provides one additional equation – the so-called *interaction equation* – in comparison to the κ-Poincaré model, which relates the endpoints of interacting worldlines in spacetime, and we demonstrated that this equation cannot be invariant under κ-Poincaré boosts. Hence the κ-Poincaré model is not compatible in a covariant way with the dS RLF model.

This is an important observation, for it means that we need to look for an alternative interaction equation. An alternative equation that tells us how the endpoints of worldlines behave in interactions. In Special Relativity all worldlines end in a single point, and in the RLF there is the modified interaction equation, but neither of those behave covariantly in the κ-Poincaré model. One might say therefore that the κ-Poincaré interactions are at this point defined mostly on *momentum space*, and the spacetime interpretation is not completely understood yet. We leave this issue to future research, but in appendix B we make some suggestions.
Another point of interest for future research is the question if it is possible in the κ-Poincaré model to redefine the physical momenta of particles in such a way that the composition law of momenta becomes trivial. This idea is inspired by the difference in definition of momentum between Newtonian Mechanics and Special Relativity. In the first case the momentum of a particle is given simply by mv, with m the mass of the particle and v its velocity. In the latter case, however, one defines the momentum as γmv, where the Newtonian momentum is multiplied by the γ factor. As a result of this redefinition the momentum composition law in Special Relativity becomes trivial. Perhaps there exists another redefinition of momentum such that the κ-Poincaré composition law becomes trivial as well. A brief discussion on this topic is provided in appendix C.

Appendices

Appendix A

Geodesics from a Hamiltonian Flow

In this appendix we discuss the coincidence of the results from 1.3.1 and 1.3.2.

Suppose we are given an arbitrary (pseudo-) Riemannian manifold (M, g). There is a standard way of obtaining the geodesics on M as the trajectories generated by a Hamiltonian. The phase space will be the cotangent bundle T^*M with the standard Poisson bracket. As Hamiltonian we take $H = \frac{1}{2}g^{\mu\nu}p_\mu p_\nu$. We will see that the trajectories generated by this Hamiltonian are precisely the geodesics on M.

Massive Particles

From a physics point of view, when dealing with massive particles it is often be more convenient to take a slightly different Hamiltonian, $H = \frac{1}{2m}g^{\mu\nu}p_\mu p_\nu$, because then p^μ will be the physical momentum. This Hamiltonian yields the equations

$$\ddot{x}^\rho + \Gamma^\rho_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0, \qquad p^\mu = m\dot{x}^\mu, \tag{A.1}$$

which we will prove in a moment. The first of these we recognize as the geodesic equation on spacetime, and the second one shows that p^μ is indeed the physical

momentum. Well, there is actually another thing we must require in order for p to be the physical momentum. Namely, we need to make sure that the tangent vector to the curve is properly normalized, i.e., $g_{\mu\nu}p^\mu p^\nu = m^2$, or equivalently, $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 1$ (for massive particles). So Hamilton's equations together with the dispersion relation (i.e., normalization) determine uniquely a geodesic and its physical momentum.
The derivation is as follows. Hamilton's equations are

$$\dot{x}^\rho = \{x^\rho, H\} = \frac{1}{m}g^{\rho\nu}p_\nu \qquad \left(= \frac{1}{m}p^\rho\right), \tag{A.2}$$

$$\dot{p}_\sigma = \{p_\sigma, H\} = -\frac{1}{2m}p_\mu p_\nu \partial_\sigma g^{\mu\nu}. \tag{A.3}$$

Taking another derivative of the first equation, we obtain

$$\ddot{x}^\rho = \frac{1}{m}\partial_\alpha g^{\rho\nu}\dot{x}^\alpha p_\nu + \frac{1}{m}g^{\rho\nu}\dot{p}_\nu = \frac{1}{m}\partial_\alpha g^{\rho\nu}\dot{x}^\alpha p_\nu + \frac{1}{m}g^{\rho\nu}\left(-\frac{1}{2m}p_\mu p_\alpha \partial_n u g^{\mu\alpha}\right) \tag{A.4}$$

$$= \frac{1}{m}\left(-g^{\rho\rho'}g^{\nu\nu'}\partial_\alpha g_{\rho'\nu'}\right)\dot{x}^\alpha p_\nu - \frac{1}{2m^2}g^{\rho\nu}p_\mu p_\alpha\left(-g^{\mu\mu'}g^{\alpha\alpha'}\partial_\nu g_{\mu'\alpha'}\right) \tag{A.5}$$

$$= -g^{\rho\rho'}\partial_\alpha g_{\rho'\nu'}\dot{x}^\alpha\dot{x}^{\nu'} + \frac{1}{2}g^{\rho\nu}\partial_\nu g_{\mu'\alpha'}\dot{x}^{\mu'}\dot{x}^{\alpha'} \tag{A.6}$$

$$= -\frac{1}{2}g^{\rho\sigma}\left(2\partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\mu\nu}\right)\dot{x}^\mu\dot{x}^\nu \tag{A.7}$$

$$= -\frac{1}{2}g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right)\dot{x}^\mu\dot{x}^\nu \tag{A.8}$$

$$= -\Gamma^\rho_{\mu\nu}\dot{x}^\mu\dot{x}^\nu. \tag{A.9}$$

Here we have used that

$$\partial_\sigma g^{\mu\nu} = -g^{\mu\mu'}g^{\nu\nu'}\partial_\sigma g_{\mu'\nu'} \tag{A.10}$$

in going to the second line.
Note that had we used $H = \frac{1}{2}g^{\mu\nu}p_\mu p_\nu$ instead of $H = \frac{1}{2m}g^{\mu\nu}p_\mu p_\nu$, we would have gotten the set of the equations

$$\ddot{x}^\rho + \Gamma^\rho_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0, \qquad p^\mu = \dot{x}^\mu, \tag{A.11}$$

which, analogously, we must complete by adding the normalization condition $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 1$. In this alternative setting, the physics is identical, but p^μ is not the physical momentum; it is the four-velocity of the particle. To get the physical momentum we must multiply by the mass m.

Massless Particles

For massless particles we clearly cannot use the Hamiltonian $H = \frac{1}{2m}g^{\mu\nu}p_\mu p_\nu$, since we cannot divide by $m = 0$. Hence we will use the original one, $H = \frac{1}{2}g^{\mu\nu}p_\mu p_\nu$. As we have seen above, this leads to Hamilton's equations

$$\ddot{x}^\rho + \Gamma^\rho_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0, \qquad p^\mu = \dot{x}^\mu, \tag{A.12}$$

which we supplement in this case by the normalization condition $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$ (which makes sure the particle is massless, i.e., travels on a null geodesic). Here again, p^μ is not the physical momentum. So how do we get the physical momentum? We cannot just multiply by m as above, since then the momentum would vanish. The definition of the momentum of a massless particle in General Relativity is $p^\mu_{\text{ph}} = \mathrm{d}x^\mu/\mathrm{d}\tilde{\lambda}$, where $\tilde{\lambda}$ is a certain affine parameter. In the case of massive particles one takes $\tilde{\lambda} = \tau$, the proper time, but for massless particles this does not work since $\tau = 0$. Instead, $\tilde{\lambda}$ has to be determined experimentally in each separate case. The only thing we know is that, since $\tilde{\lambda}$ is an affine parameter, it has to be given as $\tilde{\lambda} = a\lambda + b$ (for some $a \neq 0$, b) in terms of the (arbitrary) original affine parameter λ. Then $p^\mu_{\text{ph}} = \mathrm{d}x^\mu/\mathrm{d}\tilde{\lambda} = (\mathrm{d}x^\mu/\mathrm{d}\lambda)(\mathrm{d}\lambda/\mathrm{d}\tilde{\lambda}) = \dot{x}^\mu/a$ and hence the normalization condition is equivalent to

$$g_{\mu\nu}p^\mu_{\text{ph}}p^\nu_{\text{ph}} = 0. \tag{A.13}$$

Now the interesting fact that makes the 'symmetry approach' of the de Sitter spacetime (section 1.3.2) work just as the above Hamiltonian approach and hence as the conventional General Relativity approach (section 1.3.1), is that the phase space representation of quadratic Casimir (1.43) of the de Sitter algebra (which we used as Hamiltonian in 1.3.2), is proportional to the Hamiltonian $\frac{1}{2}g^{\mu\nu}p_\mu p_\nu$ that generates geodesics according to the discussion above, which is easily checked. This is true in 1+1 dimensions it turns out that it holds other dimensions as well.

Appendix B

Alternatives to the RLF boundary equation

B.1 The Missing Link

As we have seen, the κ-Poincaré model is not completely covariantly compatible with Relative Locality framework (RLR) because of the fact that the 'interaction equation' of the RLF,

$$x_I^\mu(\lambda_0) = z^\nu \frac{\partial \mathscr{K}_\nu}{\partial p_\mu^I}, \tag{B.1}$$

is not invariant under boosts (in the case of the κ-Poincaré composition law) or under translations (in the case of the alternative composition law of chapter 7. Hence something is still missing from the κ-Poincaré model, namely a criterion that tells us if a given combination of particles, in a given spacetime configuration, can interact, i.e., form a vertex. In Special Relativity the criterion is simply that the worldlines meet in a single point z^μ; in the Relative Locality framework the criterion is that the endpoints of the worldlines are given by the interaction equation above for some 'interaction coordinate' z^μ. So when can particles interact in the κ-Poincaré model (with either composition law)? In the best scenario one would be able to find an alternative such equation for the κ-Poincaré model. In the following we describe a possible route to obtaining such an equation. We

leave it for further study to find explicit solutions. (Although we actually do obtain an explicit invariant equation in this way, it turns out that is not physically acceptable, as we will see.)

B.2 Finding an Alternative Criterion

B.2.1 Requirements

First we need to think carefully what it is that we actually want out of the criterion.

1. Most importantly, given the (supposed) endpoints of the worldlines of a set of incoming particles, the criterion should tell us whether that set of particles can form a vertex[1], allowing for a (possibly nonlocal) interaction.

2. Secondly, it must also determine the starting points of the worldlines of the resulting particles.

3. We would also like to enforce the (weak) principle of relative locality. This does not mean that we need the usual equation from the Relative Locality framework, but it means that we require that there always exists an (inertial) observer for which a given interaction vertex is local, i.e., all particles interact in a single point. We have called this the *weak* principle of relative locality; the *strong* principle of relative locality then states that the *weak* principle holds and, additionally, that if an (inertial) observer is local to one of the interacting particle's endpoints, then it must be local to all other interacting particle endpoints and hence local to the interaction vertex.

4. (Optional.) The *strong* principle of relative locality should hold.

Note that the requirements (3) and (4) tacitly assume that the theory is invariant under a well-defined set of transformations that relate (inertial) observers. Note also that it is very convenient, in the light of these requirements, to specify an

[1]Note that a vertex here does not mean that the worldlines must actually meet. The $x^\mu(\lambda_0)$ of the Relative Locality framework do not meet in general either.

interaction using a parameter z^μ, so that if the endpoint of the worldline of one particle is known, we will be able to determine z^μ, and from there we will be able to determine the allowed endpoints for the other particles. The criteria for an interaction in Special Relativity trivially satisfies all four requirements. The RLF does so if and only if $(\partial \mathscr{K} / \partial p)$ is an invertible matrix (assuming, of course, invariance under a set of observer-relating symmetry transformations).

B.2.2 Derivation

A simple ansatz for the criterion is the equation[2]

$$x^\mu(\lambda_0) = z^\nu \, T(p)_\nu{}^\mu, \tag{B.2}$$

for some interaction coordinate z^μ and some matrix (not necessarily a tensor) $T(p)_\nu{}^\mu$ that may depend on the momentum p of the particle. This ansatz satisfies all of the requirements outlined above if and only if $T(p)$ is invertible. Given this ansatz it is easy to derive the requirement that T needs to satisfy, which we will now do. (From now on we will omit the λ_0.) It is easy to see that for any T this equation can be made invariant under translations $z^\mu \to z^\mu + a^\mu$, by simply defining the translation behavior of the x^μ in a suitable way. Hence here we will focus on boosts. The requirement that the criterion is invariant under boosts can be written as

$$\tilde{z}^\nu \, T(\tilde{p})_\nu{}^\mu = \tilde{x}^\mu = x^\alpha \frac{\partial p_\alpha}{\partial \tilde{p}_\mu} = z^\nu \, T(p)_\nu{}^\alpha \frac{\partial p_\alpha}{\partial \tilde{p}_\mu} \tag{B.3}$$

which is equivalent (assuming that $T(p)_\nu{}^\mu$, as a matrix, is invertible, which we may do because it *has* to be invertible in order to satisfy our requirements) to the following transformation rule for z^μ, written in matrix notation,

$$\tilde{z} = z \, T(p) \, \frac{\partial p}{\partial \tilde{p}} \, T(\tilde{p})^{-1}. \tag{B.4}$$

The required condition on T follows from the requirement that this transformation rule be well-defined. Note that if z is the interaction coordinate corresponding to a vertex in which (possibly among others) the momenta p and q participate, then the same logic yields the transformation

[2]Note that the RLF interaction equation is not of this form because in that case T does not only depend on p but on all momenta in the vertex.

$$\tilde{z} = z\, T(q)\, \frac{\partial q}{\partial \tilde{q}}\, T(\tilde{q})^{-1}. \tag{B.5}$$

Since there is only *one* z^μ that corresponds to this vertex, the two expressions for $\tilde{z}$ must coincide. The required condition on T is hence that

$$\boxed{T(p)\, \frac{\partial p}{\partial \tilde{p}}\, T(\tilde{p})^{-1} \text{ is independent of } p.} \tag{B.6}$$

We now would like to find matrices $T(p)$ that satisfy this condition.
A general class of solutions is given by $T(p)_\nu{}^\mu = \frac{\partial \phi_\nu}{\partial p_\mu}$, where ϕ is any $\mathbb{R}^N$-valued function (in N dimensions) on momentum space with *constant, linear* transformation behavior under boosts, i.e., $\tilde{\phi}_\mu = A_\mu{}^\nu \phi_\nu$ with A a constant matrix. (Here $\tilde{\phi} = \phi(\tilde{p})$.) In this case

$$\tilde{T} = \frac{\partial \tilde{\phi}}{\partial \tilde{p}} = \frac{\partial \tilde{\phi}}{\partial p} \frac{\partial p}{\partial \tilde{p}} \tag{B.7}$$

and hence we see that

$$T(p)\, \frac{\partial p}{\partial \tilde{p}}\, T(\tilde{p})^{-1} = \frac{\partial \phi}{\partial p}\, \frac{\partial p}{\partial \tilde{p}}\, \left(\frac{\partial \tilde{\phi}}{\partial \tilde{p}}\right)^{-1} = \frac{\partial \phi}{\partial \tilde{\phi}} \frac{\partial \tilde{\phi}}{\partial p}\, \frac{\partial p}{\partial \tilde{p}}\, \left(\frac{\partial \tilde{\phi}}{\partial p} \frac{\partial p}{\partial \tilde{p}}\right)^{-1} = \frac{\partial \phi}{\partial \tilde{\phi}} = A^{-1}. \tag{B.8}$$

which is independent of p and hence solves our problem. In particular, if ϕ is an *invariant*, it will do the job. Now the first candidate for ϕ that comes to mind in the case of the κ-Poincaré model is of course a Casimir. The only problem is that we need ϕ to have N components, with N the dimension of momentum space/spacetime, whereas the Casimir C has only one. So in 1+1 dimensions, one invariant is not enough. One possibility is to define $\phi^\mu(p) = C(p)$ for each μ, but this leads to the issue that the matrix T has rank one, so it does not satisfy our requirements. (And it also means that the endpoint of a particle with given momentum p can lie only on a one-dimensional subspace of Minkowski space, which is not what we want.) To find, for the case of $1+1$ dimensions, the second component of ϕ, we might try to find a second function, besides the Casimir, which is invariant under boosts. We leave this for further study.

Appendix C

Redefinition of Physical Momentum

An alternative to looking for a κ-deformed momentum addition law when going from the $\kappa = \infty$ limiting case (i.e., Special Relativity) to finite κ is to take the same approach as when going from Galilean physics to Special Relativistic physics, namely *redefining* the physical momenta[1] precisely such that the momentum addition law is trivial. With the κ-Poincaré law of the previous section we can achieve this, to first order in $1/\kappa$, by for instance defining the physical momenta as

$$p_0^{\text{ph}} = p_0 + \frac{p_1^2}{2\kappa}, \quad p_1^{\text{ph}} = p_1 + \frac{p_0 p_1}{\kappa}. \tag{C.1}$$

In these coordinates, the momentum conservation law becomes trivial,

$$p^{\text{ph}} \oplus q^{\text{ph}} = p^{\text{ph}} + q^{\text{ph}}, \tag{C.2}$$

and moreover, the κ-Poincaré dispersion relation becomes trivial in terms of the physical momenta as well:

$$m^2 = (p_0^{\text{ph}})^2 - (p_1^{\text{ph}})^2 = \eta^{\mu\nu} p_\mu^{\text{ph}} p_\nu^{\text{ph}}, \tag{C.3}$$

[1] Namely, $p \to \gamma p$

and even the momentum space metric becomes the trivial Minkowski one

$$g_{\mu\nu}(p^{\mathrm{ph}}) = \eta_{\mu\nu}. \tag{C.4}$$

What's more, the κ-Poincaré boosts are just the standard Minkowski boost in these coordinates.

The reason for all of this is that it turns out that actually the so defined physical momenta correspond, to first order in $1/\kappa$, to two of the three Minkowski embedding coordinates of momentum space. And we already knew that everything (well, almost everything) was trivial in that description. The difference, however, is that we now interpret the change of coordinates as a change of chart on our momentum manifold, instead of as an embedding in a higher dimensional space. To first order in $1/\kappa$ this is clearly possible, as the coordinate change given above is invertible to this order. But it might be the case that this first order transformation is extendable to a diffeomorphism to all orders in ξ, in which case at some point (i.e., at some order) it would stop resembling the embedding in Minkowski space, but it might still make the equations trivial.

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