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Exact Gravitational Waves in Finsler Gravity

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Introduction

The field of Finsler gravity studies extensions of general relativity (GR) based on Finsler geometry, a natural generalization of Riemannian geometry. This poster shows our novel exact gravitational wave solutions, to be published in [1], to the Finslerian field equations proposed in [2].

Finsler Geometry

Finsler geometry is often said to be *Riemannian geometry without the quadratic restriction*, referring to the fact that the line element $\mathrm{d}s = F(x, \mathrm{d}x^\mu)$ is given by a minimally constrained so-called **Finsler function** F, not necessarily the square root of a Riemannian quadratic form.

Example. (Randers metric)

One of the most important examples of a Finsler function. It satisfies the axioms of a Finsler function iff* $g^{\mu\nu}b_\mu b_\nu < 1$.

$$F = \underset{\substack{\uparrow \\ \textbf{Riemannian} \\ \textbf{metric}}}{\sqrt{g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu}} + \underset{\substack{\uparrow \\ \textbf{1-form}}}{b_\mu \mathrm{d}x^\mu}$$

*This holds in positive definite signature. For Lorentzian Randers metrics the situation is more involved.

~~Levi-Civita connection~~ The Cartan non-linear connection

The canonical connection on a Finsler space is the Cartan non-linear connection. It is *the unique torsion-free, metric-compatible, homogeneous (non-linear) connection on the tangent bundle*, and as such it generalizes the Levi-Civita connection. As in Riemannian geometry, geodesics (length-extremizing curves) in Finsler geometry coincide with autoparallels of the canonical connection (curves that parallel-transport their own tangent vector). In Finsler gravity these are the curves along which test particles travel.

Sometimes the Cartan non-linear connection turns out to be linear. In that case we say that the geometry is of **Berwald** type. For Riemannian spaces (which are always Berwal) the canonical connection coincides with the levi-Civita connection.

Exact Solutions to the Field Equations

With the following theorem [1] we introduce a large class of exact solutions to Finsler gravity. Some of the already known solutions fall into this class, for instance the ones found in [4] and [3].

Theorem 1.

Let (M, F) be any Finsler spacetime with Finsler function F constructed from only a Lorentzian metric $g = g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu$ and a 1-form $\beta = b_\mu \mathrm{d}x^\mu$. If (M, g) is a vacuum solution to Einstein gravity and b_μ is covariantly constant, then (M, F) is an exact solution to Finsler gravity of Berwald type.

pp-waves

The vacuum solutions to Einstein gravity admitting a covariantly constant 1-form are precisely the **pp-waves**. Hence theorem 1 provides a way to lift Lorentzian pp-wave spacetimes in GR to proper Finsler pp-wave spacetimes. The simplest pp-wave metric is given by

$$\mathrm{d}s^2 = -2\mathrm{d}u\mathrm{d}v + H(u, x, y)\,\mathrm{d}u^2 + \mathrm{d}x^2 + \mathrm{d}y^2,$$

where u, v are lightcone coordinates. This is a vacuum solution in general relativity if and only if $(\partial_x^2 + \partial_y^2)H = 0$.

Example. (Finsler pp-waves of Randers type)

An example of an **exact gravitational wave in Finsler gravity**, following **Theorem 1**, is provided by the following Randers metric

$$F = \sqrt{-2\mathrm{d}u\mathrm{d}v + H(u, x, y)\mathrm{d}u^2 + \mathrm{d}x^2 + \mathrm{d}y^2} + \mathrm{d}u,$$

under the condition that $(\partial_x^2 + \partial_y^2)H = 0$.

Finsler Gravity

The term Finsler gravity applies to theories of gravity obtained by enlarging the allowed class of geometries in General Relativity from Lorentzian to (Lorentz-)Finsler geometries. To make such a theory mathematically precise one would have to specify the exact definition of a Lorentz-Finsler geometry and specify a field equation that extends Einstein's field equation, which is beyond the scope of this poster. In the literature today the most investigated proposal is the Finsler gravity framework by Pfeifer and Wohlfarth. We will only cover the important special case of Berwald spacetimes.

Berwald Spacetimes

A Finsler space is said to be of Berwald type if the Cartan non-linear connection is in fact linear.

For Berwald spaces the **vacuum field equation** reduces to [3]

$$\left({}^F g^{\mu\nu} - \frac{3}{F^2}y^\mu y^\nu\right) R_{\mu\nu} = 0\,, \qquad (1)$$

in terms of the Ricci tensor

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} = \partial_\lambda \Gamma^\lambda_{\nu\mu} - \partial_\nu \Gamma^\lambda_{\lambda\mu} + \Gamma^\lambda_{\sigma\lambda}\Gamma^\sigma_{\nu\mu} - \Gamma^\lambda_{\sigma\nu}\Gamma^\sigma_{\lambda\mu}$$

of the corresponding *linear* connection, and the fundamental tensor ${}^F g_{\mu\nu} = \partial_\mu \partial_\nu \left(\frac{1}{2}F^2\right)$, which plays the role of a generalized metric tensor. Clearly the vanishing of the Ricci tensor $R_{\mu\nu} = 0$ (which is reminiscent of Einstein's vacuum field equations) is a sufficient condition for having a solution to Eq. (1), and for Lorentzian manifolds this condition is also necessary, providing the correct limit to GR.

[1] S. Heefer, L. Florack, and A. Fuster, "Finsler (α, β)-gravitational waves," *Upcoming*.

[2] C. Pfeifer and M. N. R. Wohlfarth, "Finsler geometric extension of Einstein gravity," *Phys. Rev.* **D85** (2012) 064009.

[3] A. Fuster, C. Pabst, and C. Pfeifer, "Berwald spacetimes and very special relativity," *Phys. Rev.* **D98** (2018), no. 8, 084062.

[4] A. Fuster and C. Pabst, "Finsler pp-waves," *Phys. Rev.* **D94** (2016), no. 10, 104072.

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