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Randers pp-waves

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Based on joint work [1] with Christian Pfeifer and Andrea Fuster

Introduction

Finsler gravity is an extension of general relativity (GR) based on Finsler geometry [2], a generalization of Riemannian geometry that naturally appears e.g. in the study of quantum gravity phenomenology [3]. This poster exhibits our novel Randers pp-wave solutions [1, 4] to the Finslerian vacuum field equations proposed in [5]. These generalize the well-known pp-wave spacetimes in GR.

Finsler geometry

Finsler geometry can be characterized as *Riemannian geometry without the quadratic restriction*, referring to the fact that the line element $\mathrm{d}s = F(x, \mathrm{d}x)$ is given by a minimally constrained so-called **Finsler metric** F, not necessarily the square root of a Riemannian quadratic form.

Example. (Randers metric)

One of the most prevalent examples of a Finsler metric.

$$F = \underbrace{\sqrt{a_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu}}_{\text{Riemannian metric}} + \underbrace{b_\mu\,\mathrm{d}x^\mu}_{\text{1-form}} \tag{1}$$

Finsler gravity

The term Finsler gravity applies to theories of gravity obtained by enlarging the allowed class of geometries in GR from (pseudo-)Riemannian to Finsler geometries of Lorentzian signature. We consider here the framework by Pfeifer and Wohlfarth [5]. The Finslerian extension of Einstein's field equation in vacuum is given by

$$\mathrm{Ric} - \frac{F^2}{3} g^{\mu\nu} R_{\mu\nu} - \frac{F^2}{3} g^{\mu\nu}\left(\bar\partial_\mu \dot S_\nu - S_\mu S_\nu + \nabla_{\delta_\mu} S_\nu\right) = 0, \tag{2}$$

where we omit the exact definitions of all symbols because of space limitations. The key takeaway here is that the equation is incredibly complex. Nevertheless, by restricting to Randers metrics (1) of Berwald type (definition follows) we can establish the following result, reminiscent of the situation in GR.

Lemma 1.

A Berwald-Randers spacetime is a vacuum solution in Finsler gravity if and only if the Finsler Ricci tensor vanishes,

$$R_{\mu\nu} = 0$$

[1] S. Heefer, C. Pfeifer, and A. Fuster, "Randers pp-waves," *Phys. Rev.* **D104** (Jul, 2021) 024007.
[2] D. Bao, S.-S. Chern, and Z. Shen, *An introduction to Finsler-Riemann geometry*. Springer, New York, 2000.
[3] F. Girelli, S. Liberati, and L. Sindoni, "Planck-scale modified dispersion relations and Finsler geometry," *Phys. Rev.* **D75** (2007) 064015.
[4] S. Heefer and A. Fuster, "Finsler gravitational waves of (α,β)-type and their observational signature," 2023. preprint: arXiv:2302.08334.
[5] C. Pfeifer and M. N. R. Wohlfarth, "Finsler geometric extension of Einstein gravity," *Phys. Rev.* **D85** (2012) 064009.

Berwald spacetimes and the Cartan non-linear connection

The canonical connection on a Finsler space is the Cartan non-linear connection. It is *the unique torsion-free, metric-compatible, homogeneous connection on the tangent bundle*, and as such it generalizes the Levi-Civita connection. It is, however, not necessarily linear. As in Riemannian geometry, geodesics (length-extremizing curves) in Finsler geometry coincide with autoparallels of the canonical connection (curves that parallel-transport their own tangent vector). In Finsler gravity such (timelike) curves describe the motion of freely-falling test particles.

If the Cartan non-linear connection is in fact linear then we say that the geometry is of **Berwald** type. For Riemannian spaces (which are always Berwald) the canonical connection coincides with the Levi-Civita connection.

Exact solutions to the field equations

With the following novel theorem we characterize all Berwald-Randers vacuum solutions to Finsler gravity, by relating them to vacuum solutions to the classical Einstein field equation.

Theorem 1.

Let $F = \alpha + \beta$ be any Randers metric of Berwald type, constructed from a Lorentzian metric $\alpha = \sqrt{a_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu}$ and a 1-form $\beta = b_\mu \mathrm{d}x^\mu$. Then F is an exact vacuum solution to Finsler gravity **if and only if** $a_{\mu\nu}$ is a vacuum solution to GR and $\nabla_\mu b_\nu = 0$, i.e. b_μ is covariantly constant with respect to $a_{\mu\nu}$.

pp-waves

It can be shown that any (nontrivial) vacuum solution to GR admitting a covariantly constant 1-form, as in the theorem, is a CCNV (covariantly constant null vector) spacetime, also commonly refered to as a **pp-wave**. And in particular, the 1-form b_μ *must be null.* Any vacuum pp-wave in GR can be written in the form

$$\mathrm{d}s^2 = -2\mathrm{d}u\mathrm{d}v + H(u,x,y)\,\mathrm{d}u^2 + \mathrm{d}x^2 + \mathrm{d}y^2,$$

where H can loosely be interpretted as the amplitude of a wave traveling on a Minkowski background, and $\mathrm{d}u$ is the associated covariantly constant null 1-form. This is a vacuum solution in GR if and only if $(\partial_x^2 + \partial_y^2)H = 0$.

Conclusion

Thus, Theorem 1 tells us the following:

A Berwald-Randers spacetime is a vacuum solution to Finsler gravity if and only if it is composed of a GR vacuum pp-wave and its covariantly constant null 1-form, i.e.

$$\mathrm{d}s = F = \sqrt{-2\mathrm{d}u\mathrm{d}v + H(u,x,y)\mathrm{d}u^2 + \mathrm{d}x^2 + \mathrm{d}y^2} + \mathrm{d}u,$$

where $(\partial_x^2 + \partial_y^2)H = 0$.