The Schwarzschild Solution

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These lecture notes contain the material for lectures 8 and 9 of the course *General Relativity* (3ERX0), corresponding roughly to chapter 9 of the book [1].

Contents

1 Introduction

When Einstein completed his general theory of relativity in 1915, the German physicist and astronomer Karl Schwarzschild had been serving as a Lieutenant of the German artillery in World War I. Due to an autoimmune skin disease, however, he was now stuck in a hospital at the Russian front. In this period he studied Einstein's results and only about a month after Einstein had published his field equations Schwarzschild managed to find the very first exact solution (other than flat Minkowski space), now called the *Schwarzschild solution*. It describes the gravitational field surrounding a static, spherical mass distribution, such as a planet, a star or even a black hole. Only a few months later Schwarzschild died.

In Chapter 1 of these lecture notes we will derive, using the symmetries of the problem and Einstein's field equations, the form of the Schwarzschild metric.

In Chapter 2 we will discuss some of the phenomenological implications of the Schwarzschild solution: the trajectories of freely falling objects and light rays (i.e. geodesics of the Schwarzschild geometry), redshift, and gravitational time-dilatation.

2 The Schwarzschild Solution

2.1 The general static spherically symmetric line-element

In this chapter we will derive the form of the gravitational field surrounding a stationary and spherically symmetric mass distribution like a star or a planet. It is to be expected that the symmetries of the mass distribution carry over to the gravitational field, so before we turn to the Einstein field equations (EFEs) the first thing we will do is make use of these symmetries to reduce the degrees of freedom in the metric. In fact we will find that the metric tensor will be completely determined by only 2 functions of a single variable. Only then will we use the EFEs in order to determine the form of these two functions. The relevant symmetries to impose are *staticity*, which is essentially time-invariance, and *spherical symmetry*, which needs a particularly careful treatment in the general relativistic setting.

2.1.1 Staticity: invariance under time translation and reversal

Given a stationary mass distribution, we expect the geometry of spacetime to be constant in ‘time’. But given what we know about the relativity of (space and) time, we should be very precise about what we actually mean by this. The proper statement to make is that we require that there exists *some* coordinate system ($x^0 = t, x^1, x^2, x^3$) such that the line-element is invariant under time translations $t \to t - a$. As you might expect, and as proven in Exercise 1, this is equivalent to the property that in these coordinates, $g_{\mu\nu}$ does not depend on the time coordinate t. The line-element will thus in general be of the form

$$\mathrm{d}s^2 = g_{tt}(x)\mathrm{d}t^2 + 2g_{ti}(x)\mathrm{d}t\mathrm{d}x^i + h_{ij}(x)\mathrm{d}x^i\mathrm{d}x^j, \qquad x = (x^1, x^2, x^3).$$

A spacetime that satisfies this property of time-independence is said to be *stationary*.

We also expect the geometry to be invariant under time-reversal[1]. In the line-element above, all terms are time-reversal invariant except the $\mathrm{d}t\mathrm{d}x^i$ term, which acquires a minus sign under the transformation. Hence $\mathrm{d}s^2$ can only be invariant under $t \to -t$ if $g_{ti}(x)\mathrm{d}t\mathrm{d}x^i = -g_{ti}(x)\mathrm{d}t\mathrm{d}x^i$, implying that $g_{ti}(x)\mathrm{d}t\mathrm{d}x^i = 0$. Thus all $\mathrm{d}t\mathrm{d}x^i$ terms must vanish. A spacetime that is time-reversal invariant as well as stationary is said to be

[1] This has to do with the fact that our mass is assumed to be non-rotating. If it were rotating, a change $t \to -t$ would flip the rotation direction of the mass and might therefore also flip the gravitational field in some way. For a non-rotating body we don’t expect this to happen.

static. Our results so far can therefore be summerized as follows.

In a static spacetime the line-element must be of the following general form

$$\mathrm{d}s^2 = g_{tt}(x)\mathrm{d}t^2 + h_{ij}(x)\mathrm{d}x^i\mathrm{d}x^j, \qquad x = (x^1, x^2, x^3). \tag{1}$$

2.1.2 Spherical symmetry

Spherical symmetry in the general relativistic context is a slippery concept because we are working in an *a priori* arbitrarily curved space with arbitrary coordinates. What do we even mean by spherical symmetry if we are not in Euclidean space? Below we give an intuitive approach to the concept[2].

The first step is to view our space as foliated by spherical shells, or simply spheres. We know that the line-element on a sphere is given by[3]

$$\mathrm{d}s^2 = -r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right) =: -r^2\mathrm{d}\Omega^2,$$

where θ, ϕ are the usual angular coordinates. The coordinate r is usually interpreted as the radius of the sphere, however, this interpretation is based on the conventional view that the sphere is embedded in Euclidean three-dimensional space! In our present situation the spheres will be embedded in some general, as of yet unknown geometry, which might (and will) turn out to be curved. So we should not interpret r as a conventional radius as we usually would. Its sole purpose at this stage is to characterize all the different possible spherical surfaces. We will come back to its actual physical meaning later. This foliation of space into spheres then gives us a preferred set of spatial coordinates r, θ and ϕ, in addition to the preferred time coordinate t that we already obtained from staticity, so that Eq. (1) now reads

$$\mathrm{d}s^2 = g_{tt}(r,\theta,\phi)\mathrm{d}t^2 + g_{rr}(r,\theta,\phi)\mathrm{d}r^2 + 2g_{r\theta}(r,\theta,\phi)\mathrm{d}r\mathrm{d}\theta + 2g_{r\phi}(r,\theta,\phi)\mathrm{d}r\mathrm{d}\phi + \mathrm{d}\tilde{\Omega}^2,$$

[2] The more mathematically oriented mind might be happy to know that a perfectly precise definition of spherical symmetry can be formulated with the help of advanced Lie group theory and differential geometry. Following [2], a (pseudo-)Riemannian manifold is said to be spherically symmetric if the rotation group $SO(3)$ acts on it as isometry (sub)group, in such a way that each orbit (i.e. the collection of points that can be reached from a given starting point by applying rotations) is a spacelike, simply-connected, complete two-dimensional surface. Given this definition a theorem can be proven rigorously, stating that any spherically symmetric line-element must indeed be of the form (2) in certain local coordinates. That is, the rigorous mathematical approach is completely consistent with ours. Details can be found in [2, 3, 4].

[3] The minus sign comes form the fact that we are using a $(+,-,-,-)$ spacetime signature, leading to a $(-,-,-)$ spatial signature, whereas usually if we consider a sphere in $\mathbb{R}^3$ we use $(+,+,+)$ signature. This gives an overall minus sign.

where $\mathrm{d}\tilde{\Omega}^2$ represents the terms involving $\mathrm{d}\theta^2$, $\mathrm{d}\theta\mathrm{d}\phi$ and $\mathrm{d}\phi^2$. Note that by setting $t =$ const and $r =$ const (so that $\mathrm{d}t = \mathrm{d}r = 0$) we find that the spherical shell at the given value of t and r, which we argued above should satisfy $\mathrm{d}s^2 = -r^2\mathrm{d}\Omega^2$, has the line-element $\mathrm{d}s^2 = \mathrm{d}\tilde{\Omega}^2$, so that we can immediately deduce that $\mathrm{d}\tilde{\Omega}^2 = -r^2\mathrm{d}\Omega^2$.
By spherical symmetry we now simply mean that the line-element should be invariant under rotations[4]. Consider first the $\mathrm{d}r\mathrm{d}\theta$ and $\mathrm{d}r\mathrm{d}\phi$ terms. We can rewrite

$$g_{r\theta}(r,\theta,\phi)\mathrm{d}r\mathrm{d}\theta + g_{r\phi}(r,\theta,\phi)\mathrm{d}r\mathrm{d}\phi = \mathrm{d}r\left(g_{r\theta}(r,\theta,\phi)\mathrm{d}\theta + g_{r\phi}(r,\theta,\phi)\mathrm{d}\phi\right) =: \mathrm{d}r\,\omega.$$

Here $\mathrm{d}r$ is spherically invariant, but ω is a covector (field). Just like a vector defines a preferred direction, a covector defines a preferred hyperplane through the origin, which breaks spherical invariance just as well as a preferred direction would. Therefore the term $\mathrm{d}r\omega$ cannot be spherically invariant and has to vanish. The metric thus reduces to

$$\mathrm{d}s^2 = g_{tt}(r,\theta,\phi)\mathrm{d}t^2 + g_{rr}(r,\theta,\phi)\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2.$$

Last but not least, spherical symmetry now requires that g_{tt} and g_{rr} cannot depend on θ and ϕ, because $\mathrm{d}t$ and $\mathrm{d}r$ do not change under rotations. In summary, we can conclude the following:

In a static, spherically symmetric spacetime the metric must be of the form

$$\mathrm{d}s^2 = A(r)\mathrm{d}t^2 - B(r)\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right). \tag{2}$$

We have thus reduced our problem to the determination of just two functions, $A(r)$ and $B(r)$, of the single coordinate r.

2.2 The exterior solution

We have shown above that the most general static spherically symmetric line-element is of the form (2). To find $A(r)$ and $B(r)$ we turn to Einstein's field equations. Note that $A(r) > 0$ and $B(r) > 0$ because we want t to be a timelike coordinate[5] and the signature of the metric to be Lorentzian. Outside of the mass distribution the energy-momentum tensor vanishes, so we can use the following form of the Einstein field equations in vacuum

$$R_{\mu\nu} = 0.$$

[4]Rotations act on the preferred coordinates (t, r, θ, ϕ) in the usual way; in particular they transform only the angles and leave r and t unchanged.

[5]What we mean by this is that the 4-velocity of a worldline that goes only in the t-direction should be timelike.

Given the line element (2) one can calculate the the Christoffel symbols, the Riemann tensor and finally the Ricci tensor. We will not reproduce this straightforward but tedious calculation here, but in the end one finds that the only non-trivial components of the Ricci tensor, and hence the Einstein equations, are given by

$$\begin{aligned}
R_{tt} &= -\frac{A''}{2B} + \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rB} = 0, \\
R_{rr} &= \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rB} = 0, \\
R_{\theta\theta} &= \frac{1}{B} - 1 + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right) = 0, \\
R_{\phi\phi} &= R_{\theta\theta}\sin^2\theta = 0.
\end{aligned}$$

Note that the last equation is redundant as it is implied by the third. Note also that we can write

$$0 = \frac{A}{B}R_{tt} + R_{rr} = -\frac{1}{r}\left(\frac{A'}{A} + \frac{B'}{B}\right) = -\frac{1}{rAB}\left(A'B + AB'\right) = -\frac{1}{rAB}\left(AB\right)',$$

so that

$$AB = \alpha$$

for some constant α. Substituting $B = \alpha/A$ in the θ-equation it reduces to

$$0 = R_{\theta\theta} = \frac{A}{\alpha} - 1 + \frac{rA}{2\alpha}\left(2\frac{A'}{A}\right) = \frac{A}{\alpha} - 1 + \frac{rA'}{\alpha} = \frac{1}{\alpha}\left(rA\right)' - 1$$

so that $rA = \alpha(r + k)$, with k another integration constant[6] and we can conclude that

$$A(r) = \alpha\left(1 + \frac{k}{r}\right), \qquad B(r) = \left(1 + \frac{k}{r}\right)^{-1}.$$

This leads to the line-element

$$\mathrm{d}s^2 = \alpha\left(1 + \frac{k}{r}\right)\mathrm{d}t^2 - \left(1 + \frac{k}{r}\right)^{-1}\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right), \tag{3}$$

[6]Note that the actual integration constant is $\tilde{k} = \alpha k$, so that the notation with k is only possible in general as long as $\alpha \neq 0$. It is justified here, however, because we will see that indeed $\alpha \neq 0$.

where the constants α and k are still to be determined.

Asymptotically, as $r \to \infty$, this reduces to

$$\mathrm{d}s^2 = \alpha\, \mathrm{d}t^2 - \mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right). \tag{4}$$

The requirement that sufficiently far away from the source the gravitational 'force' is negligible and hence that spacetime should reduce to flat Minkowski space in this limit yields[7] $\alpha = c^2$. More generally, by expanding to first order in k/r we note that for large values of r the metric becomes approximately equal to

$$\mathrm{d}s^2 = c^2\left(1 + \frac{k}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{k}{r}\right)\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right). \tag{5}$$

Recall (see Hobson Eq. 7.8) that in the weak field limit, in order to have the correct Newtonian limit, we must have $g_{00} = 1 + 2\Phi/c^2 = 1 - 2GM/c^2 r$. But note that presently we have an additional factor c^2, as we are working with the coordinate t instead of the coordinate $x^0 = ct$. Hence $g_{tt} = c^2\left(1 - 2GM/c^2 r\right)$. Requiring that in the weak field limit general relativity reduces to Newtonian gravity we thus find that $k = -2GM/c^2$, and we conclude the following:

The gravitational field outside a static, spherical mass distribution is described by the Schwarzschild metric*:*

$$\mathrm{d}s^2 = c^2\left(1 - \frac{2GM}{c^2 r}\right)\mathrm{d}t^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1}\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right). \tag{6}$$

Notice that something peculiar happens at a coordinate radius $r = r_s \coloneqq \frac{2GM}{c^2}$. Here the metric seems to blow up and everything seems[8] to break down. For a typical star like our sun, however, the surface is located at a radius much larger than this so-called

[7]Another way of looking at it is that different values of α simply correspond to different choices of units in which we measure time, since we can change α by scaling the t-coordinate.

[8]However, as will be discussed later, by choosing a different set of coordinates, it is possible to describe this part of spacetime perfectly well. That is, $r = r_s$ is only a so-called coordinate singularity, not a true singularity.

Schwarzschild radius r_s. Only for *extremely* compact objects – objects with an enormous mass compared to their size – will this radius play a role. This would lead us into the realm of black holes; these will be discussed later in the course. (See Hobson chapter 11).

2.3 Birkhoff's theorem

Birkhoff's theorem, which we will not prove here, states that the Schwarzschild solution is the only spherically symmetric solution to Einstein's field equations in vacuum. Note the absence of the word 'static'. Indeed, staticity turns out to be a superfluous assumption in our derivation above; it is automatically implied by spherical symmetry. In particular, this implies that there exist no 'time-dependent' solutions. An important consequence of the theorem is that a purely radially pulsating star cannot emit gravitational radiation, because outside of this star such gravitational radiation would amount to a time-dependent spherically symmetric spacetime geometry in (approximate) vacuum, which, according to the Birkhoff's theorem, cannot be consistent with Einstein's field equations.

3 Particle Trajectories

3.1 Geodesic equations

Given the Schwarzschild line-element derived in the previous chapter, it is in principle straightforward to calculate its Christoffel symbols and consequently the geodesic equations (or one may use the method with the Euler-Lagrange equations, which is not as tedious, and it is in fact a very good exercise to do this yourself). We do not reproduce the calculation in detail here, this can be found in (Hobson §9.5). The geodesic equations for the Schwarzschild geometry read

$$\left(1-\frac{2\mu}{r}\right)\dot{t}=k, \tag{7}$$

$$\left(1-\frac{2\mu}{r}\right)^{-1}\ddot{r}+\frac{\mu c^2}{r^2}\dot{t}^2-\left(1-\frac{2\mu}{r}\right)^{-2}\frac{\mu}{r^2}\dot{r}^2-r\left(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2\right)=0, \tag{8}$$

$$\ddot{\theta}+\frac{2}{r}\dot{r}\dot{\theta}-\sin\theta\cos\theta\dot{\phi}^2=0, \tag{9}$$

$$r^2\sin^2\theta\dot{\phi}=h, \tag{10}$$

where $\mu = GM/c^2$, and k and h are constants. It turns out that all geodesics in the Schwarzschild geometry are *planar*, i.e. lie in a 2D plane through the origin. Should you be wondering why this is the case, Exercise 3 guides you through a proof. In what follows we will take the result for granted. By spherical symmetry we may then simply chose the orbital plane to be the $\theta = \pi/2$ plane. Throughout the remainder of this section we will therefore set $\theta = \pi/2$, and, unless otherwise stated, in any exercise you may do so as well. The geodesic equations then simplify to

$$\left(1-\frac{2\mu}{r}\right)\dot{t}=k, \tag{11}$$

$$\left(1-\frac{2\mu}{r}\right)^{-1}\ddot{r}+\frac{\mu c^2}{r^2}\dot{t}^2-\left(1-\frac{2\mu}{r}\right)^{-2}\frac{\mu}{r^2}\dot{r}^2-r\dot{\phi}^2=0, \tag{12}$$

$$r^2\dot{\phi}=h. \tag{13}$$

The last ingredient needed to completely describe the movement of particles and light in the Schwarzschild geometry is the correct normalization of geodesics, $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = c^2$ for massive particles or $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$ for massless particles.

In summary, in the Schwarzschild geometry we have the following equations of motion:

Schwarzschild Equations of Motion

Schwarzschild geodesics:

$$\left(1-\frac{2\mu}{r}\right)\dot{t}=k, \tag{14}$$

$$r^2\dot{\phi}=h, \tag{15}$$

$$\left(1-\frac{2\mu}{r}\right)^{-1}\ddot{r}+\frac{\mu c^2}{r^2}\dot{t}^2-\left(1-\frac{2\mu}{r}\right)^{-2}\frac{\mu}{r^2}\dot{r}^2-r\dot{\phi}^2=0, \tag{16}$$

Schwarzschild normalization equation:

$$c^2\left(1-\frac{2\mu}{r}\right)\dot{t}^2-\left(1-\frac{2\mu}{r}\right)^{-1}\dot{r}^2-r^2\dot{\phi}^2=\begin{cases}c^2 & \text{(massive particles)}\\ 0 & \text{(massless particles)}\end{cases}. \tag{17}$$

The constants k and h represent a kind of energy and angular momentum, respectively, as will be explored in Exercise 1.

As a remark we note that in a lot of cases, Eq. (16) is in fact redundant and the two other geodesic equations (14),(15) together with the somewhat simpler normalization equation, Eq. (17), suffice to completely describe the motion. (See Exercise 2 for a proof.)

3.2 The effective potential

Let us do the following calculation for massive and massless particles simultaneously by writing $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu=\varepsilon c^2$ with $\varepsilon=1$ and $\varepsilon=0$ for massive and massless particles, respectively. Substituting Equations (14) and (15) in (17) and multiplying by $(1-2\mu/r)$ yields

$$c^2k^2-\dot{r}^2-\left(1-\frac{2\mu}{r}\right)\frac{h^2}{r^2}=\left(1-\frac{2\mu}{r}\right)\varepsilon c^2,$$

or

$$\dot{r}^2 + \left(1 - \frac{2GM}{c^2 r}\right)\frac{h^2}{r^2} - \frac{2GM}{r}\varepsilon = c^2(k^2 - \varepsilon),$$

which we recognize as being very similar to the equation governing a Newtonian particle in a 1-dimensional potential $V(r)$, namely we have

$$\boxed{\frac{1}{2}\dot{r}^2 + V(r) = \frac{c^2(k^2-\varepsilon)}{2} = \text{const}, \qquad V(r) = -\frac{GM}{r}\varepsilon + \frac{h^2}{2r^2} - \frac{GMh^2}{c^2r^3}. \quad (18)}$$

This equation reveals an enormous amount of information about the possible orbits in the Schwarzschild geometry. Just as in Newtonian mechanics, another derivative (in the present case with respect to the affine parameter) reveals that[9]

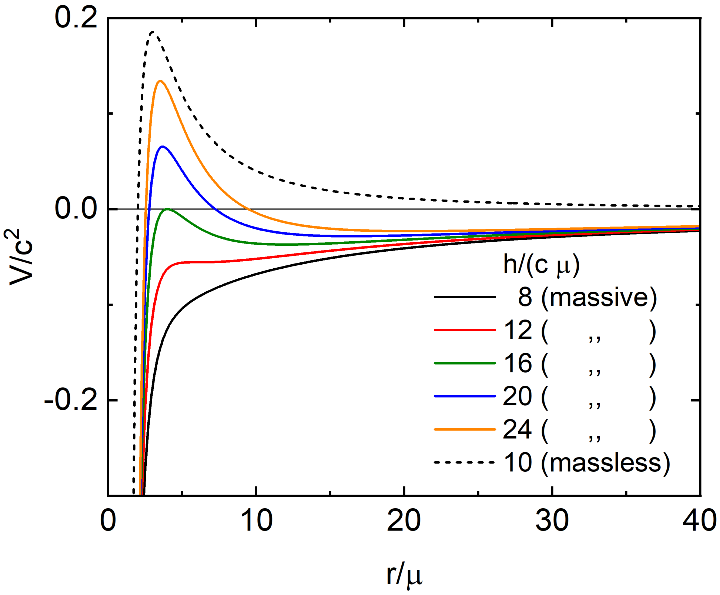
$$\ddot{r} = -V'(r). \tag{19}$$

This means nothing else than that particles feel an effective 'force' towards the direction of smaller effective potential. The shape of the potential is shown in Figure 3.2 for different values of the parameters. These plots already give us a good qualitative idea of the possible orbits. Something peculiar we can infer immediately from the local maxima is that *even for massless particles* there is apparently a circular orbit! The union of all such orbits forms a sphere, called the *photon sphere*. In principle then, if you could get to this radius, it would be possible to see the back of your own head by looking around the massive body. However, as shown in Exercise 7, the coordinate radius of the photon sphere is given by $r = 3\mu$, so for a typical star like our sun, for which $3\mu \ll R_\odot$, this orbit is irrelevant, as it would lie underneath the sun's surface. We will discuss circular orbits of massive particles, and by (very good) approximation, massive bodies like planets moving around a star, in the next section.

3.3 Circular orbits of massive bodies

For circular orbits we have $r = \text{const}$, so from the viewpoint of the effective potential we are interested in the points where $V'(r) = 0$. For stable circular orbits we additionally

[9] Actually, you might object that this argument only works when $\dot{r} \neq 0$. And you would be right. Nevertheless (19) always holds, also when $\dot{r} = 0$, which can be seen by substituting the $\dot{t}$ and $\dot{\phi}$ equation and (18), in the $\ddot{r}$ geodesic equation, eliminating $\dot{t}, \dot{\phi}$, and $\dot{r}$, respectively.



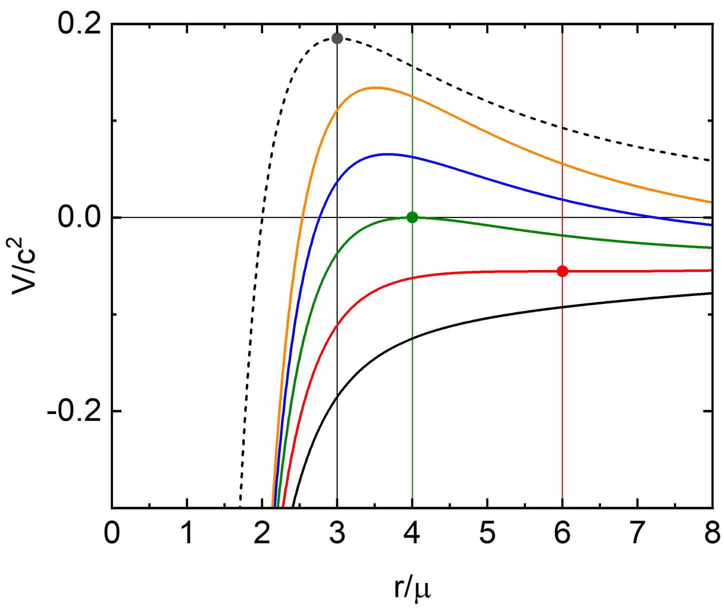


Figure 1: These plots, made by Peter Bobbert, show the shape of the effective potental for different values of $h/c\mu$ and for massless as well as massive particles. In the massive case the exact shape depends on the value of this parameter. For instance, if the value is less than 12 there is no local minimum, and hence no circular orbits, whereas for larger values, such orbits do exist. In the massless case, on the other hand, different values of $h/c\mu$ do not lead to fundamentally different shapes. In fact the massless potential is always a constant multiple of the one for $h/c\mu = 10$ shown in the figure.

require that $V''(r) > 0$, i.e. stable circular orbits occur precisely at the local minima of the potential. The form of $V(r)$ is such that it has at most one local minimum[10]. The equation $V'(r) = 0$ can be solved for h, giving

$$h^2 = \frac{\mu c^2 r^2}{r - 3\mu}.$$

By positivity of the LHS this shows that all circular orbits must satisfy $r > 3\mu$, and that if indeed $r > 3\mu$ there is precisely 1 orbit (since a given value of r completely fixes $\dot{\phi}$ and $\dot{t}$, fixing the orbit, at least up to initial conditions.) Substituting the expression for h into $V''(r)$ we find that

$$V''(r) = \frac{\mu c^2 (r - 6\mu)}{r^3(r - 3\mu)}.$$

Consequently:

For $r > 6\mu$ circular orbits are stable and for $3\mu < r \leq 6\mu$ circular orbits are not stable. For $r < 3\mu$ no circular orbits exist and any massive free-falling particle crossing this radius will keep falling to smaller and smaller radii.

Now let's find the remaining components of the 4-velocity for circular orbits. Using

$$\frac{1}{2}\dot{r}^2 + V(r) = c^2(k^2 - 1)/2$$

we find that

$$k = \frac{1 - 2\mu/r}{\sqrt{1 - 3\mu/r}}.$$

This fixes

$$\dot{t} = k\left(1 - \frac{2\mu}{r}\right)^{-1}.$$

Similarly, h, fixed by r, fixes

$$\dot{\phi} = \frac{h}{r^2}.$$

[10]To see this, e.g. write $r = 1/u$, then V is simply a third order polynomial. which has at most one local minimum. Note that a local minimum in u corresponds to local minimum in r.

Hence the four-velocity of a massive particle in a circular orbit is given by

$$u^\mu = \left(\frac{k}{1 - \frac{2\mu}{r}}, 0, 0, \frac{h}{r^2} \right).$$

You can check that we indeed have $u_\mu u^\mu = c^2$, as required for a massive object.

Apart from *stability* of orbits we can also talk about *boundedness*. An orbit is said to be bounded if the particle in the orbit can never reach (spatial) infinity. Looking at the effective potential equation, this is the case when $\epsilon = c^2(k^2 - 1)/2 < \lim_{r\to\infty} V(r) = 0$. That is, $k < 1$. Looking at the expression for k above, this implies that circular bound orbits are the ones for which $r > 4\mu$. Hence a circular orbit with $3\mu < r \leq 4\mu$ is unstable and not bound, so a particle in such an orbit might, due to even the slightest perturbation in its orbit, fly off to infinity.

3.4 Photon trajectories

As shown in Exercise 7 there is a single circular orbit for massless particles (in the equatorial and hence any plane), forming the *photon sphere*. This unstable orbit is located at $r = 3\mu$. The radial trajectories of photons are treated in Exercise 6.

3.5 Gravitational Redshift

The last topic to be discussed in these notes is gravitational redshift, to be precise: redshift due to the gavitational field of a massive spherical body such as a star or a planet. We consider two stationary observers, an emitter $\mathcal{O}_E$ at radius r_E and a receiver $\mathcal{O}_R$ at radius r_R. and a photon traveling, along a geodesic, from emitter to receiver. We would like to find out what, if any, is the difference between the photon energy (equivalently, photon frequency or wavelength) the two observers measure.

In general the energy of a photon (or of any particle, really) as measured by an observer is given by

$$E = g_{\mu\nu} u^\mu p^\nu, \tag{20}$$

where u^μ is the four-velocity of the observer and p^μ is the four-velocity of the photon. This follows from the equivalence principle: in a local intertial frame that is (at some fixed point along the observer's worldline) freely falling with the observer, the laws of physics should reduce to those of special relativity. Hence the metric in these coordinates is simply the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, and since the observer is at rest in these coordinates (at least at the fixed point on his worldline, not necessarily away from it because the observer itself need not be freely falling), we moreover have $u^\mu = (u^0, 0, 0, 0)$. Consequently the formula for the energy reduces in these coordinates to $E = \eta_{00} u^0 p^0 = cp^0$, which is indeed exactly the familiar formula for the energy in special relativity. So in the local inertial coordinates, Eq. (20) is correct. But since it is a tensorial equation with no free indices on the LHS, the result is *independent of the coordinate system.* And hence this formula for the energy is correct in *any* coordinate system.

Now let's return to our two observers and let's consider either of them. Since the observer is stationary in the Schwarzschild geometry, we have $u^\mu = (u^0, 0, 0, 0)$ (in our standard Schwarzschild coordinates). The normalization equation then tells us that $u^0 = g_{tt}^{-1/2}$. Hence

$$E = g_{tt}^{1/2} p^0.$$

The photon momentum is given by $p^\mu = \mathrm{d}x^\mu/\mathrm{d}\sigma$ in terms of some affine parameter[11] σ, so that $p^0 = \mathrm{d}t/\mathrm{d}\sigma$. Now recall that, by the geodesic equation (11), $g_{tt}\frac{\mathrm{d}t}{\mathrm{d}\sigma} = k$ is constant along the photon's worldline. Substituting all this into the formula for the energy, we find that

$$E = g_{tt}^{-1/2} k$$

and we can compare the two energies the two observers measure as

$$\frac{E_R}{E_E} = \frac{g_{tt}^{-1/2}(R)k}{g_{tt}^{-1/2}(E)k} = \sqrt{\frac{1-2\mu/r_E}{1-2\mu/r_R}} \approx 1 - \frac{\mu}{r_E} + \frac{\mu}{r_R}.$$

Note that this result is independent of the exact photon worldline (it does not have to be radial, for instance). The last approximate equality is valid for values of $r_{E,R}$ that

[11] Recall that for any *massive* particle we can always parameterize the worldline using the proper time τ, and thus the 4-velocity is given, by definition, by $\mathrm{d}x^\mu/\mathrm{d}\tau$ and the 4-momentum by $m\mathrm{d}x^\mu/\mathrm{d}\tau$. For a massless particle there is no such thing as proper time, so we cannot use these same definitions. We can still define a photon 4-velocity by $\mathrm{d}x^\mu/\mathrm{d}\sigma$, where σ is any affine parameter, but note that this depends on what affine parameter that we choose. Similarly, the 4-momentum can be defined as $p^\mu = \alpha\mathrm{d}x^\mu/\mathrm{d}\sigma$ for some constant α, but by absorbing α into a new affine parameter $\lambda = \sigma/\alpha$, we can also write $p^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda$ in terms of the new affine parameter. What this affine parameter looks like cannot be inferred from the photon worldline alone, but is determined by the photon's physical energy and momentum.

are very large compared to μ. On Earth this is an excellent approximation, since the Schwarzschild radius of the Earth is about a centimeter. If moreover the emitter is relatively close to the receiver (again relative to μ), $r_E = r_R + \delta r$, we have approximately

$$\frac{E_R}{E_E} = 1 + \frac{\mu}{r_E^2}\delta r.$$

Plugging in some values for the earth,

$$\begin{aligned} G &= 6.674 \cdot 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}, \\ M &= M_\oplus = 5.972 \cdot 10^{24}\,\mathrm{kg}, \\ r_E &= r_\oplus = 6378\,\mathrm{km}, \\ \delta r &= 22.5\,\mathrm{m}, \end{aligned}$$

we find that

$$E_R/E_E - 1 = 2.45 \cdot 10^{-15}.$$

In other words

$$E_R - E_E = 2.45 \cdot 10^{-15} E_E.$$

This is an *extremely* small difference in energy. Still it is of relevance and can in fact be measured! In the Pound-Rebka experiment in 1960 an emitter was placed at the top of the Harvard physics building, sending out γ-photons through radioactive decay, and a receiver at the bottom, made of atoms of ^{57}Fe that can essentially only absorb photons of *exactly* the emitted frequency. No absorption was detected, as expected from gravitational redshift. However, subsequently, Pound and Rebka gave the emitter a velocity, so that the Doppler-shift would, in theory, exactly cancel out the gravitational redshift. And indeed, now there was absorption, thus confirming the predictions of GR.

4 Some Exercises

Exercise 1. Symmetries = conserved quantities.
This exercise aims to illustrate how symmetries and conserved quantities are intimately linked, leading to a deep physical interpretation of the (at first sight) mysterious constants k and h appearing in the geodesic equations for the Schwarzschild geomety.

Consider a spacetime with metric $g_{\mu\nu}(t, x^i)$. By time-translation invariance we mean that the line-element is invariant under time translations, that is, under coordinate transformations of the form $t \to \tilde{t} = t - a$, with a constant.

(a) Show that time-translation invariance is equivalent to the property that $g_{\mu\nu} = g_{\mu\nu}(x^i)$, i.e. $g_{\mu\nu}$ does not depend on t. *Hint: keep carefully track of the argument of $g_{\mu\nu}$ when applying the transformation!*

(b) Consider a massive particle of mass m moving along a geodesic $x^\mu(\tau)$. Show that the covariant form of the 4−momentum p_μ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\tau} p_\mu = \frac{m}{2} \left(\partial_\mu g_{\alpha\beta}\right) \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}.$$

(c) Combine (a) and (b) to show that if a spacetime is time-translation invariant then the covariant energy[12] $E \equiv p_t$ is conserved along geodesics.

(d) Generalize your findings in (c) to translation-invariance in an arbitrary coordinate, i.e. not just the t coordinate.

(e) Apply your results to the Schwarzschild metric (6) and provide a physical interpretation of the constants k and h appearing in the geodesic equations for the Schwarzschild geometry.

Exercise 2. Show that the normalization equation (17) together with the two geodesic equations (14), (15) imply that the remaining geodesic equation (16) is automatically satisfied, except possibly when $\dot{r} = 0$. *Hint: employ the effective force equation, Eq.* (19).

Exercise 3. Particle trajectories are planar. This exercise serves to show that without loss of generality we may always choose to work in the equatorial plane $\theta = \pi/2$ when discussing geodesics. (Hence in contrast to most other exercises we initially do **not** set $\theta = \pi/2$ here.)

[12]More precisely, this is the energy as measured by a stationary observer at infinity, which would be a good exercise to check from the general energy formula as well.

(a) Show that the quantity

$$L^2 = p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}$$

is conserved along geodesics in the Schwarzschild metric, where $p_\mu = g_{\mu\nu}\dot{x}^\nu$. (We don't include the mass m here in the definition of the momentum because we also want to be able to describe massless particles) *Hint: Show that the θ-geodesic equation, Eq. (9), implies $dL^2/d\tau = 0$.*

(b) Check that we may write L^2 as

$$L^2 = r^4\dot{\theta}^2 + \frac{h^2}{\sin^2\theta}$$

and show that at $\theta = \pi/2$, $\dot{\theta} = 0$, this expression attains a local minimum $L^2 = h^2$, in the sense that for any other choice of values (close to these values), L^2 will necessarily become larger.

(c) Consider a freely falling particle with initial conditions $\theta = \pi/2$, $\dot{\theta} = 0$. Conclude that this particle will remain forever within the equatorial plane $\theta = \pi/2$.

(d) Use the previous results together with spherical symmetry to argue that **in the Schwarzschild geometry *any* freely-falling particle trajectory is planar**, i.e. lies in some plane through the origin.

Exercise 4. Hobson exercise 9.12. A particle is dropped from rest at a coordinate radius $r = R$ in the Schwarzschild geometry. Obtain an expression for the 4-velocity of the particle in (t, r, θ, ϕ) coordinates when it passes coordinate radius r.

Exercise 5. Hobson exercise 9.11, slightly modified. A particle is dropped from rest at infinity in the Schwarzschild geometry.

(a) Find the amount of proper time $\Delta\tau$ it takes for the particle to travel from a coordinate radius r_0 (with $r_0 > r_s$) to another radius r, where $r_s = 2\mu$ is the Schwarzschild radius. *Hint: use the result from the previous exercise. You may use the standard integral*

$$\int \frac{x^{3/2}}{x-1}\mathrm{d}x = \frac{2}{3}\sqrt{x}(x+3) + \ln\left|\frac{\sqrt{x}-1}{\sqrt{x}+1}\right|$$

(b) Find the amount of coordinate time Δt it takes for the particle to travel from r_0 to r.

(c) Conclude that it takes an infinite amount of coordinate time for the particle to reach $r = r_s$, while from the point of view of the particle itself, it will do so in a finite amount of proper time.

Exercise 6. Show that radial photon trajectories satisfy

$$\begin{aligned} c\Delta t &= r + 2\mu \ln \left| \frac{r}{2\mu} - 1 \right| + \text{const}, \qquad &\text{(outgoing photon)} \\ c\Delta t &= -r - 2\mu \ln \left| \frac{r}{2\mu} - 1 \right| + \text{const}. \qquad &\text{(ingoing photon)} \end{aligned}$$

Exercise 7. Use the effective potential to prove that there exists a unique circular photon orbit (in the equatorial plane). Show that this orbit is located at $r = 3GM/c^2$ and is unstable.

5 Solutions to the Exercises

Solution to Exercise 1.

(a) Applying a coordinate transformation $t \to \tilde{t} = t - a$ (supplemented by $\tilde{x}^i = x^i$ for proper bookkeeping), the metric in the new coordinates is given by

$$\tilde{g}_{\mu\nu}(\tilde{t}, \tilde{x}^i) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(\tilde{t}, \tilde{x}^i) = g_{\mu\nu}(\tilde{t}, \tilde{x}^i) = g_{\mu\nu}(t - a, x^i).$$

Time-invariance is the statement that this transformed metric is the same as the original one, i.e. $g_{\mu\nu}(t - a, x^i) = g_{\mu\nu}(t, x)$. Since a can take on any value this just means that $g_{\mu\nu}$ does not depend on t.

(b) Letting the overdot denote $\mathrm{d}/\mathrm{d}\tau$ we start by writing out

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau} p_\mu &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{\mu\nu} p^\nu\right) = m \frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{\mu\nu}\dot{x}^\nu\right) \\ &= m\left(\dot{g}_{\mu\nu}\dot{x}^\nu + g_{\mu\nu}\ddot{x}^\nu\right) \\ &= m\left(\partial_\rho g_{\mu\nu}\dot{x}^\rho\dot{x}^\nu + g_{\mu\nu}\ddot{x}^\nu\right),\end{aligned}$$

where in the last line we have used the chain rule:

$$\dot{g}_{\mu\nu} = \frac{\mathrm{d}}{\mathrm{d}\tau} g_{\mu\nu} = \frac{\partial}{\partial x^\rho} g_{\mu\nu}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau} = \partial_\rho g_{\mu\nu}\dot{x}^\rho. \tag{21}$$

x^μ is a geodesic, so it satisfies the geodesic equation

$$\ddot{x}^\nu = -\Gamma^\nu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta.$$

Plugging this into our expression, and using also the definition of the Christoffel symbols, yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau} p_\mu &= m\left(\partial_\rho g_{\mu\nu}\dot{x}^\rho\dot{x}^\nu - g_{\mu\nu}\Gamma^\nu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta\right) \\ &= m\left(\partial_\rho g_{\mu\nu}\dot{x}^\rho\dot{x}^\nu - \frac{1}{2} g_{\mu\nu} g^{\nu\sigma}\left(\partial_\alpha g_{\sigma\beta} + \partial_\beta g_{\sigma\alpha} - \partial_\sigma g_{\alpha\beta}\right)\dot{x}^\alpha\dot{x}^\beta\right) \\ &= m\left(\partial_\rho g_{\mu\nu}\dot{x}^\rho\dot{x}^\nu - \frac{1}{2}\delta^\sigma_\mu\left(\partial_\alpha g_{\sigma\beta} + \partial_\beta g_{\sigma\alpha} - \partial_\sigma g_{\alpha\beta}\right)\dot{x}^\alpha\dot{x}^\beta\right),\end{aligned}$$

where we have used the fact that $g_{\mu\nu}g^{\nu\sigma} = \delta^\sigma_\mu$. The Kronecker δ ensures that the whole second part of the expression is non-vanishing only when the dummy index σ is equal to μ, so we may set $\sigma = \mu$, leading to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}p_\mu &= m\left(\partial_\rho g_{\mu\nu}\dot{x}^\rho\dot{x}^\nu - \frac{1}{2}\left(\partial_\alpha g_{\mu\beta} + \partial_\beta g_{\mu\alpha} - \partial_\mu g_{\alpha\beta}\right)\dot{x}^\alpha\dot{x}^\beta\right) \\ &= m\left(\partial_\rho g_{\mu\nu}\dot{x}^\rho\dot{x}^\nu - \frac{1}{2}\partial_\alpha g_{\mu\beta}\dot{x}^\alpha\dot{x}^\beta - \frac{1}{2}\partial_\beta g_{\mu\alpha}\dot{x}^\alpha\dot{x}^\beta + \frac{1}{2}\partial_\mu g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta\right).\end{aligned}$$

After relabeling the dummy indices ($\rho \to \alpha, \nu \to \beta$ in the first term and $\beta \leftrightarrow \alpha$ in the third term) this reads

$$\frac{\mathrm{d}}{\mathrm{d}\tau}p_\mu = m\left(\partial_\alpha g_{\mu\beta}\dot{x}^\alpha\dot{x}^\beta - \frac{1}{2}\partial_\alpha g_{\mu\beta}\dot{x}^\alpha\dot{x}^\beta - \frac{1}{2}\partial_\alpha g_{\mu\beta}\dot{x}^\alpha\dot{x}^\beta + \frac{1}{2}\partial_\mu g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta\right)$$

and we observe that the first three terms cancel each other out, leaving us with the desired formula

$$\frac{\mathrm{d}}{\mathrm{d}\tau}p_\mu = \frac{m}{2}\partial_\mu g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta.$$

(c) If the spacetime is time-translation invariant then, by (a), $g_{\mu\nu}$ does not depend on t, hence $\partial_t g_{\alpha\beta} = 0$ and by setting $\mu = t$ in (b) we find therefore that $\mathrm{d}p_t/\mathrm{d}\tau = 0$. Hence p_t is conserved along geodesics.

(d) There is nothing special about the t coordinate. What your results in (a-c) show is that if the line-element is invariant under translations of a certain coordinate, then the corresponding component of the covariant four-momentum is a conserved quantity.

(e) The Schwarzschild metric is invariant under t-translations, and under ϕ-translations, so the covariant energy and ϕ-component of momentum, which we interpret as an angular momentum, are conserved quantities. Using the Schwarzschild metric we compute that

$$\begin{aligned}E = p_t = g_{tt}p^t = mg_{tt}\dot{t} = mc^2\left(1 - \frac{r_s}{r}\right)\dot{t} = kmc^2, \\ p_\phi = g_{\phi\phi}p^\phi = -mr^2\sin^2\theta\dot{\phi} = -mh.\end{aligned}$$

Thus we conclude that the constants k and h are a measure of the particle's energy and ϕ-angular momentum, respectively. k and h are constants because, due to the symmetries of the Schwarzschild metric, this energy and angular momentum are conserved quantities along geodesics.

Solution to Exercise 2. As derived in the text, if we eliminate $\dot{t}$ and $\dot{\phi}$ using the corresponding geodesic equations, the normalization equation $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \varepsilon c^2$ leads to (18), i.e. $\dot{r}^2/2 + V(r) = \text{const}$, another derivative of which yields $(\ddot{r} + V'(r))\dot{r} = 0$. Hence, we see that $\ddot{r} = -V'(r)$, which is Eq. (19). This argument only works when $\dot{r} = 0$, and although the resulting equation is true for $\dot{r} = 0$ as well (as explained in the text), the geodesic equation (16) that we are currently trying to *derive* is needed to be able to infer this, so we cannot use this fact right now and hence we have to assume for the moment that $\dot{r} = 0$. Substituting $V'(r)$ for its explicit expression and then eliminating ε using the normalization equation, Eq. (17), leads to

$$0 = \ddot{r} + \frac{\mu c^2 k^2}{Ar^2} - \frac{\mu \dot{r}^2}{Ar^2} - \frac{h^2}{r^3}A,$$

where we have written $A = (1 - 2\mu/r)$. Eliminating the constants k and h using the two geodesic equations (14), (15) yields

$$0 = \ddot{r} + \frac{\mu c^2 A}{r^2}\dot{t}^2 - \frac{\mu}{Ar^2}\dot{r}^2 - Ar\dot{\phi}^2,$$

which, after multiplying by A^{-1} yields the desired remaining geodesic equation (16).

Solution to Exercise 3.

(a) First note that

$$\begin{aligned}
p_\theta &= g_{\theta\theta}p^\theta = -r^2\dot{\theta}, \\
p_\phi &= g_{\phi\phi}p^\phi = -r^2\sin^2\theta\dot{\phi} = -h, \\
\dot{p}_\theta &= -2r\dot{r}\dot{\theta} - r^2\ddot{\theta}, \\
\dot{p}_\phi &= 0.
\end{aligned}$$

Then we compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}L^2 = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) &= 2p_\theta\dot{p}_\theta + \frac{2p_\phi\dot{p}_\phi}{\sin^2\theta} - \frac{2p_\phi^2}{\sin^3\theta}\cos\theta\dot{\theta} \\
&= -2r^4\dot{\theta}\left(\ddot{\theta} + \frac{2}{r}\dot{\theta}\dot{r} - \sin\theta\cos\theta\dot{\phi}^2\right),
\end{aligned}$$

where we recognize the term in parenthesis as one of the geodesic equations, Eq. (9). As the particle travels on a geodesic this term must vanish, so that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}L^2 = 0$$

and L^2 is conserved along the worldline of the particle

(b) The rewriting of L^2 is just substituting the expressions for p_θ and p_ϕ. Now note that both terms in in L^2 are nonnegative. The first term is minimal for $\dot{\theta} = 0$ and the second term is minimal for $\theta = \pi/2$ or $\theta = 3\pi/2$. If $\dot{\theta} \neq 0$, the first term will grow; and if $\theta \neq \pi/2$ but still close to $\pi/2$, the second term will grow. So for all other values of θ and $\dot{\theta}$, close to the given values, L^2 will necessarily become larger.

(c) The particle starts out with L^2 attaining its minimal value. But we showed in (a) that L^2 is conserved, so L^2 will remain at its minimal value. If the particle were to move out of the equatorial plane, θ would have to divert from $\pi/2$. However, we have argued in (b) that when this happens, L^2 becomes larger, which cannot happen because L^2 is conserver. Hence the particle is constrained to the equatorial plane.

(d) Consider an arbitrary particle trajectory with arbitrary initial conditions. By performing a rotation we may choose our coordinate system such that the initial conditions of the particle become $\theta = \pi/2, \dot{\theta} = 0$. By (c) then, the particle stays in the equatorial plane in these coordinates and the trajectory is planar.

Solution to Exercise 4. The two relevant equations here are the normalization equation (17) and the t-geodesic equation (14),

$$c^2 = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = c^2 \left(1 - \frac{2\mu}{r}\right)\dot{t}^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} \dot{r}^2, \qquad \dot{t} = k\left(1 - \frac{2\mu}{r}\right)^{-1}.$$

(Note that because of the statement 'is dropped from rest' we can immediately infer that the particle must be massive, not massless.) At $r = R$ we have $\dot{r} = 0$ and hence

$$\dot{t}^2 = \left(1 - \frac{2\mu}{R}\right)^{-1},$$

which fixes the constant k as

$$k = \left(1 - \frac{2\mu}{R}\right)\dot{t} = \left(1 - \frac{2\mu}{R}\right)^{1/2}.$$

Then at coordinate radius r we find that

$$\dot{t} = k\left(1 - \frac{2\mu}{r}\right)^{-1} = \left(1 - \frac{2\mu}{R}\right)^{1/2}\left(1 - \frac{2\mu}{r}\right)^{-1}$$

and

$$
\begin{aligned}
\dot{r}^2 &= -c^2\left(1-\frac{2\mu}{r}\right) + c^2\left(1-\frac{2\mu}{r}\right)^2 \dot{t}^2 \\
&= -c^2\left(1-\frac{2\mu}{r}\right) + c^2\left(1-\frac{2\mu}{R}\right) \\
&= 2\mu c^2\left(\frac{1}{r}-\frac{1}{R}\right).
\end{aligned}
$$

Therefore

$$
\dot{r} = -\sqrt{2\mu c^2\left(\frac{1}{r}-\frac{1}{R}\right)},
$$

where the minus sign comes from the fact (due to $h = \dot{\phi}/r^2 = 0$) that $\ddot{r} = -V'(r) = -GM/r^2 < 0$, so that the particle feels an effective force 'downward'. Hence the 4-velocity at coordinate radius r reads

$$
\dot{x}^\mu = (\dot{t}, \dot{r}, 0, 0) = \left(\left(1-\frac{2\mu}{R}\right)^{1/2}\left(1-\frac{2\mu}{r}\right)^{-1}, -\sqrt{2\mu c^2\left(\frac{1}{r}-\frac{1}{R}\right)}, 0, 0\right).
$$

Solution to Exercise 5. Taking the limit $R \to \infty$ in the previous exercise we see that the 4-velocity of the particle is given by

$$
\dot{x}^\mu = (\dot{t}, \dot{r}, 0, 0) = \left(\left(1-\frac{2\mu}{r}\right)^{-1}, -\sqrt{\frac{2\mu c^2}{r}}, 0, 0\right).
$$

In particular,

$$
\begin{aligned}
\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\tau} &= -\sqrt{\frac{2\mu c^2}{r}} \\
-\int\sqrt{\frac{r}{2\mu c^2}}\mathrm{d}r &= \int \mathrm{d}\tau \\
\frac{2}{3}\sqrt{\frac{r_0^3}{2\mu c^2}} - \frac{2}{3}\sqrt{\frac{r^3}{2\mu c^2}} &= \Delta\tau,
\end{aligned}
$$

where we have integrated from r_0 to r and $\Delta\tau$ is the proper time elapsed during the fall from r_0 to r. Next, in order to find the expression $t(r)$, note that the t component of the

4-velocity says that

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \left(1 - \frac{2\mu}{r}\right)^{-1}, \qquad \frac{\mathrm{d}\tau}{\mathrm{d}t} = \left(1 - \frac{2\mu}{r}\right).$$

Then

$$\begin{aligned}
\frac{\mathrm{d}t}{\mathrm{d}r} &= \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}r} \\
&= -\frac{2}{3}\left(1 - \frac{2\mu}{r}\right)^{-1} \frac{\mathrm{d}}{\mathrm{d}r}\sqrt{\frac{r^3}{2\mu c^2}} \\
&= -\left(1 - \frac{2\mu}{r}\right)^{-1} \sqrt{\frac{r}{2\mu c^2}}.
\end{aligned}$$

Hence, writing $r = 2\mu x$,

$$\begin{aligned}
\Delta t &= -\int \left(1 - \frac{2\mu}{r}\right)^{-1} \sqrt{\frac{r}{2\mu c^2}}\mathrm{d}r \\
&= -\frac{2\mu}{c}\int \left(1 - \frac{1}{x}\right)^{-1} \sqrt{x}\mathrm{d}x \\
&= -\frac{2\mu}{c}\int \frac{x^{3/2}}{x-1}\mathrm{d}x \\
&= -\frac{2\mu}{c}\left[\frac{2}{3}\sqrt{x}(x+3) + \ln\left|\frac{\sqrt{x}-1}{\sqrt{x}+1}\right|\right] \\
&= -\frac{1}{c}\left[\frac{2}{3}\sqrt{\frac{r}{2\mu}}(r+6\mu) + 2\mu\ln\left|\frac{\sqrt{r/2\mu}-1}{\sqrt{r/2\mu}+1}\right|\right] \\
&= -\left[\frac{2}{3}\sqrt{\frac{r^3}{2\mu c^2}} + \frac{4\mu}{c}\sqrt{\frac{r}{2\mu}} + \frac{2\mu}{c}\ln\left|\frac{\sqrt{r/2\mu}-1}{\sqrt{r/2\mu}+1}\right|\right]_{r=r_0}^{r=r} \\
&= \frac{2}{3}\left(\sqrt{\frac{r_0^3}{2\mu c^2}} - \sqrt{\frac{r^3}{2\mu c^2}}\right) + \frac{4\mu}{c}\left(\sqrt{\frac{r_0}{2\mu}} - \sqrt{\frac{r}{2\mu}}\right) \\
&\quad + \frac{2\mu}{c}\ln\left|\frac{\sqrt{r_0/2\mu}-1}{\sqrt{r_0/2\mu}+1}\right|\left|\frac{\sqrt{r/2\mu}+1}{\sqrt{r/2\mu}-1}\right|.
\end{aligned}$$

Now notice that

$$\lim_{r\to r_s=2\mu}\Delta\tau = \frac{2}{3}\sqrt{\frac{r_0^3}{2\mu c^2}} - \frac{2}{3}\sqrt{\frac{r_s^3}{2\mu c^2}} < \infty,$$

$$\lim_{r\to r_s=2\mu}\Delta t = \quad \text{finite} \quad + \lim_{r\to 2\mu}\frac{2\mu}{c}\ln\left|\frac{\sqrt{r_0/2\mu}-1}{\sqrt{r_0/2\mu}+1}\right|\left|\frac{\sqrt{r/2\mu}+1}{\sqrt{r/2\mu}-1}\right| = \infty.$$

Thus it takes a finite amount of proper time for the particle to travel from r_0 to r_s. However, it takes an infinite amount of coordinate time. A static observer at a coordinate distance $r = R$ has $\Delta\tau = (1 - 2\mu/R)\,\Delta t$, so any static observer will say it takes an infinite amount of time before the particle reaches $r = r_s$, i.e., it will never reach this radius from the point of view of a static observer.

Solution to Exercise 6. A null geodesic satisfies $\mathrm{d}s^2 = 0$, hence

$$\left(1-\frac{2\mu}{r}\right)c^2\mathrm{d}t^2 = \left(1-\frac{2\mu}{r}\right)^{-1}\mathrm{d}r^2 \qquad \Rightarrow \qquad c\frac{\mathrm{d}t}{\mathrm{d}r} = \pm\left(1-\frac{2\mu}{r}\right)^{-1}.$$

Integrating, we find

$$\begin{aligned}
c\Delta t &= \pm\int\frac{1}{1-\frac{2\mu}{r}}\mathrm{d}r = \pm\int\frac{r}{r-2\mu}\mathrm{d}r \\
&= \pm\int\left(1+\frac{2\mu}{r-2\mu}\right)\mathrm{d}r \\
&= \pm\left(r + 2\mu\ln|r-2\mu|\right) + \text{const} \\
&= \pm\left(r + 2\mu\ln\left|\frac{r}{2\mu}-1\right|\right) + \text{const},
\end{aligned}$$

where in the last step we have absorbed a term $\pm 2\mu\ln|2\mu|$ into the constant.

Solution to Exercise 7. The effective potential for a massless particle, Eq. (18), reads $V(r) = \frac{h^2}{2r^2} - \frac{GMh^2}{c^2r^3}$. For a circular orbit r is constant and hence r must be located at a local extremum of this potential:

$$0 = V'(r) = -\frac{h^2}{r^3} + \frac{3GMh^2}{c^2r^4} = \frac{h^2}{r^4}\left(-r+\frac{3GM}{c^2}\right).$$

Hence $r = 3GM/c^2$. Taking another derivative, we find that $V''(r = 3GM/c^2) < 0$, hence at this value of r the effective potential does not have a local minimum (rather a local maximum) and therefore the orbit is unstable.

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